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Higher moments portfolio Optimization with unequal weights Based on Generalized Capital Asset Pricing Model with Independent and Identically Asymmetric Power Distribution

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ABSTRACT

The main criterion in investment decisions is to maximize the investors utility. Traditional capital asset pricing models cannot be used when asset returns do not follow a normal distribution. For this reason, we use capital asset pricing model with independent and identically asymmetric power distributed (CAPM-IIAPD) and capital asset pricing model with asymmetric independent and identically asymmetric exponential power distributed with two tail parameters(CAPM-AIEPD) to estimate return and risk. When the assumption of normality is violated, the first and second moments lose their efficiency in optimization and we need to use the third and fourth moments. In this article, we use a new method to estimate return and risk in abnormal distributions, and for the first time, we propose independent and identically asymmetric exponential power distributed with two tail parameters. Then, we use higher moments optimization with unequal weights to optimize portfolios. The results indicate that capital asset pricing model with independent and identically asymmetric power distributed (CAPM-IIAPD) is better than asymmetric independent and identically asymmetric exponential power distributed with two tail parameters(CAPM-AIEPD) to estimate return and risk. Adjusted Sharpe ratio in portfolio optimization in second moments are higher than others. Adjusted returns to risk in third and fourth moments in the CAPM-IIAPD model significantly differ from the CAPM-AIEPD model and have a better performance.

1 Introduction

Economic endeavors have always been threatened by numerous threats. Changes in price, Economic laws, and other influential factors in financial markets' demand and supply are the main reasons for lack of certainty and existence of risk. In recent years, international financial markets have gained greater depths and started to affect financial decisions and predictions of individuals toward the markets. Up to now, there have been countless criteria introductions to forecast, manage and calculate investment risk in financial assets that sometimes complement one another, but all of them point out the stochastic reality of risk. One such method to model financial decision making is approach of finance and mathematical modeling. [27] In financial theory, classic risk management and portfolio optimization approaches quibbled by researchers and recent financial crises have bolded the risk management's role in portfolio optimization more than ever.[7]

Dynamic risk measurement and portfolio management is one of important issues in finance world. Considering the its distribution features in real world, portfolio risk and return measurement is highly important and in financial crisis times, it yields a more accurate and reliable performance when compared to classic models. Furthermore, portfolio optimization with higher moment approach has a better performance level since it is used when asset return does not have a normal distribution.[1]

Whereas in critical situations asset return do not have a normal distribution, it is not possible to use optimization methods, such as optimization on first and second moments (MVM), that are based on normal distribution. [2]The preeminent concern of this study is to develop a solution for risk and return estimation in financial crises and optimizing portfolios. The solution for this problem is the usage of Higher Moments models in which in addition to first (mean) and second (variance) moments that are used in traditional models, higher moments are also included.[10] Therefore, with the help of return assumptions, the optimization is conducted. The two methods of Mean-Variance-Skewness-Model (MVSM) and Mean-Variance-Skewness-Kurtosis-Model are used to optimize portfolios. Since third moment models have low levels of practicality in financial crises, and to address this issue, entropy is used in optimization. [4] concluded that the usage entropy as risk index yields a higher efficiency for the portfolio. When analyzing the performance, Sharpe ratio is only valid when the data have a normal distribution since it is based on Mean-Variance theory. [3]Hence, Sharpe index can produce deceiving answers when distributions have skewness and fat tails. To tackle this, we need tools to include higher moments. In this research we aim to:

This research in the first step seeks a suitable solution to calculate the expected return and risk in real conditions and when the market is experiencing financial crises. Then, considering that the return on assets in real conditions does not follow the normal distribution, look for a solution to optimize the portfolio using higher moments (third and fourth) with two approaches MVSM and MVSKM. Finally, considering the methods of calculating the returns and risk of the optimization portfolio with unequal weights, the effect of each of them in optimizing the portfolio and the effect of each in improving the performance of the portfolio is evaluated.[22] They are goals that follow in this study include:

• Determining the best approach to achieve the expected return and portfolio risk among the CAPM, CAPM-IAPD, CAPM-IAEPD approaches

• Optimization of higher portfolio moments based on two approaches MVSM and MVSKM, according to the three approaches of calculating efficiency and comparing them

• Portfolio optimization in a time of financial crisis.

• Evaluate portfolio performance based on methods based on much higher moments

2 Theorical Framework

2.1 Review of Literature

Markowitz is the founder of a well-known structure called modern portfolio theory. The most important role of this theory is to create a risk-return portfolio framework for investors to make decisions. With a quantitative definition of investment risk, Markowitz provided a mathematical approach for investors in asset selection and portfolio management. There are many criticisms. One of the assumptions considered in this model is that the distribution of returns is normal distribution, which many studies have rejected this assumption and emphasize the abnormality of returns.According to studies Previous, asymmetry and fat tails of financial data are considered suitable assumptions for pricing financial assets.[23]

In this study, the capital asset pricing model (CAPM) is considered with the general assumption that the independent and identically error component (IIAPD) has a distribution with mean zero and variance $\sigma^2_{\varepsilon_i}$ and skewness coefficient α, which adjusts the asymmetric return distribution. In addition, the generalized Capital asset pricing model developed in this study includes the normality of returns as one of the specific modes of the model.[16]

2.1.1 Fat Tail

The term fat tail can have a variety of meanings. In fact, the fat tail is a property of a random variable, which in the common definition, indicates a higher number of observations in the random variable distribution sequence than the observations in the normal distribution sequence. Researchers have used different approaches to consider this property. One of the approaches used is mainly the use of generalized exponential functions, which include functions such as gamma variance distribution (VG), normal inverse distribution (NIG), Cauchy distribution, and Laplace distributions. In the present study, we use a generalized exponential function to find a solution in the field of financial data broad sequences. [22]

2.1.2 Asymmetric Distribution

An important fact about the return on assets is nonlinear dependency; That is, observations in which the dependence between different returns depends on market conditions. To prevent these phenomena, nonlinear models design the correlation structure in such a way that it changes according to market conditions. In contrast, linear models had the disadvantage that linear correlations in non-critical periods overestimated the correlation and in critical periods the correlation was underestimated. One way to solve this problem is to use models with independent and asymmetric error components so that in financial crises and non-compliance of distribution with normal distribution, it is possible to accurately estimate the return and risk, as well as optimize the data portfolio with the least possible error.[25] One of unrealistic assumptions of traditional models is the consideration of return distributions as normal ones, which several studies have rejected it and emphasized on the fact that asset return does not have a normal distribution. Asymmetry and fat-tail financial data tails are proper assumptions when pricing financial assets due to the results of the studies conducted in past.[9]

Numerous research was done to include more pragmatic assumptions in financial markets. The first studies conducted by [26] used a generalized t-student distribution to model capital asset pricing and used functions other than Gaussian function to research market fluctuations. Their results showed that generalized t-student had a better performance in comparison to normal distribution yet it had the fundamental problem of instability of generalized t-student distribution compared to the normal one. In his estimations, [8] ventured to use non-Gaussian distributions such as Cauchy and t-student and a combination of these function in contrast to symmetric normal distribution and found different results from the time he used normal distribution. Yet, the models he studied did not clearly mentioned their superiority over traditional models and there was not enough reason to use non-Gaussian distributions over normal distributions where also its non-Gaussian distribution had less consistency in comparison to traditional forms.[17]

In the other research, financial distress is independent variable and corporate investment behavior is dependent variable and investment opportunities is considered as interactive variable. The present research is an applied research and in terms of methodology is a correlational study. The results showed that firms with less investment opportunities tend to be less likely to invest, in addition distressed financially firms with more investment opportunities are more likely to increase investment.[15]

Some years later, [9] added new assumptions such as skewness, asymmetric interdependency, fluctuation clustering and semi-fat tail to asset return and utilizing these assumptions they compared generalized distributions. They used generalized hyperbolic distributions to estimate value at risk and demonstrated that t-student distribution with skewness has more efficiency in generalized hyperbolic distributions, and for this reason, t-student with skewness is more optimal than other distributions.

To predict the value at risk, [24] included skewness and fat tail factors and discarded fluctuation cluster and sequential interdependency from the model. Then he showed conditional mean-value at risk models have a better performance in comparison to traditional mean-variance model.

It was after this research that once again [31] used maximum-likelihood to model stretched-exponential production decline model and used this model to solve fat tails problem. In addition, he proved that this method has consistent estimation and forecast capabilities, and solved consistency problem of previous models by exponential distribution.

To solving the problem of optimal portfolio selection for asymmetric distributions of the stock returns, by putting it into a framework of a mean-variance-skewness measure. optimal solutions are explicit and are closed-form. In particular, they provide an analytical portfolio optimization solution to the exponential utility of the well-known skew-normal distribution. Therefore,Their analytical solution can be investigated in comparison to other portfolio selection rules, such as the standard mean-variance model. The new methodology is illustrated numerically.[33]

[23] applied a new approach based on Parato theory with stock return data and beta skewness in the different periods. Although Parato distribution is a suitable one for financial asset modeling, this distribution is not suitable for financial data with fat tails, returns in different periods will not have this distribution.

Using two assumptions of asymmetric and exponential asset return to the analysis, [16] tried to study the credibility degree of capital asset pricing model. Their research variables were μ and σ that respectively are used as return and variance. Investigations revealed that these two variables are not suitable for Asymmetric Power Distributions (APD) and they cannot be used to prove that asset return in capital asset pricing model is asymmetric.

To model fluctuations, [18] utilized Orthogonal Generalized Autoregressive Conditional Heteroscedasticity (OGARCH) covariance matrix and its regime changes and to analyze model's risk components, firstly, they used mean value at risk definition. Then, using OGARCH, they achieved a better efficiency in financial crisis periods compared to traditional models.

The study modeled the non-normal returns of multiple asset classes by using a multivariate truncated Lévy flight distribution and incorporating non-normal returns into the mean-conditional value at risk (M-CVaR) optimization framework. In a series of controlled optimizations, they found that both skewness and kurtosis affect the M-CVaR optimization and lead to substantially different allocations than do the traditional mean–variance optimizations.[27]

skewness is considered to measure the asymmetry of portfolio returns and a mean-risk-skewness model for portfolio selection will be proposed in uncertain environment. Here, the returns of the securities are regarded as uncertain variables which are estimated by experienced experts instead of historical data. Furthermore, the corresponding variations and crisp forms of the model are considered. To solve the proposed optimization models, a hybrid intelligent algorithm is designed.[29]

The next study, the asset return and liquidity are fuzzy variables which follow the normal possibility distributions. Liquidity is measured as the turnover rate of the asset. On the basis of possibility theory, we transform the model into a quadratic programming problem to obtain its solution. We illustrate that, in the process of investment, investors can make better use of capital by choosing their investment portfolios according to their expected return and asset liquidity.[28]

The study is intended to construct optimal portfolios and efficient frontiers with the inclusion of higher-order moments of risk. The findings show that optimized portfolios with inclusion of skewness and kurtosis are sustainable and significantly different than those from mean-variance optimized portfolios which show asymmetric and fat-tail risk.The results also endorse that induction of skewness and kurtosis affects portfolio allocation weights and expected returns.[24]

Many financial portfolios are not mean-variance-skewness-kurtosis efficient. They recommend tilting these portfolios in a direction that increases their estimated mean and third central moment and decreases their variance and fourth central moment. The advantages of tilting come at the cost of deviation from the initial optimality criterion and show the usefulness of portfolio tilting applied to the equally-weighted, equal-risk-contribution and maximum diversification portfolios in a UCITS-compliant asset allocation setting.[26]

A toolset beyond mean–variance portfolio optimization is appropriate for those instances where higher return moments might need to be taken into account, either for individual decisions or for pricing studies. Maximizing expected log surplus utility is superior for compounding returns in excess of financial obligations. Here, it is matched with a more flexible scenario representation of the investor's joint probability distribution of returns and with an agnostic optimization engine. We show simple examples based on extrapolating historical stock and bond returns and then extended using hypothetical option prices. We clarify how Black–Scholes implied volatility anomalies can arise in a portfolio context.[25]

They use Tail Mean-Variance (TMV) model, which focuses on the rare risks but high losses and usually happens in the tail of return distribution. The proposed TMV model is based on two risk measures the Tail Condition Expectation (TCE) and Tail Variance (TV) under Generalized Skew-Elliptical (GSE) distribution. They apply a convex optimization approach and obtain an explicit and easy solution for the TMV optimization problem, and then derive the TMV efficient frontier. [13]

In the other paper, They consider the estimated weights of the tangency portfolio and derive analytical expressions for the higher order non-central and central moments of these weights when the returns are assumed to be independently and multivariate normally distributed. Moreover, the expressions for mean, variance, skewness and kurtosis of the estimated weights are obtained in closed forms. Then, complement results with a simulation study where data from the multivariate normal and t-distributions are simulated, and the first four moments of estimated weights are computed by using the Monte Carlo experiment. It is noteworthy to mention that the distributional assumption of returns is found to be important, especially for the first two moments. Finally, through an empirical illustration utilizing returns of four financial indices listed in NASDAQ stock exchange, we observe the presence of time dynamics in higher moments.[19]

The research presents a mathematical model for performance-based budgeting and combines it with rolling budget for increased flexibility. The model has been designed by Chebyshev's goal programming technique with fuzzy approach. Data for calculating productivity indicators were collected from gas refineries of Iran in 2011–2015 and analysed by Excel and GAMS software. Then, the model was tested for determining the 2016 budget of those refineries. The solution of the model reduced 0.68% of the total refinery's budget compared with the actual budgets for 2016, which is higher than the annual budget of some of the companies in this group.[12]

Whereas data return does not have normal distribution, we cannot use optimization methods that are based on normal distribution such as Mean-Variance Model (MVM) and Value at risk (VaR). [11]The solution for this problem is using higher moment models that in addition to the first (mean) and second (variance) moment which are used in traditional models, the third (skewness) and fourth (Kurtosis) moments are also included. Mean-Variance-Skewness Model (MVSM) and Mean-Variance-Skewness-Kurtosis Model (MVSKM) were used to optimize estimated portfolio.[21]

2.2 Capital Asset Pricing Model(CAPM) With Independent and Identically Asymmetric Power Distribution (IIAPD) and Independent and Identically Asymmetric Exponentially Power Distribution (IAEPD)

With the assumption of borrowing and lending with a risk-free ratio, Sharpe-lintner CAPM would be as it follows:

$E(R_i) = R_f + \beta_{im} (E(R_m) - R_f)$

Here E_r is expected return of i^{th} asset, R_f risk-free return and $E(R_m)$ expected return of the market portfolio. Excess return relative to risk-free rate is described as (formula) in which Z_i shows excess return of I^{th} asset relative to risk-free return, Z_m is excess return of market portfolio and $\beta_{im} = \frac{Cov(Z_i, Z_M)}{Var(Z_M)}$ is the sensitivity coefficient return relative to market return.

The data generating process in capital asset pricing model is:

$$Z_{it} = \alpha_{im} + \beta_{im} Z_{mt} + \varepsilon_{it} \qquad \varepsilon_{it} \sim \mathrm{NID}(0, \sigma^2_{\varepsilon_i}) \tag{1}$$

According to mean-variance frame of modern portfolio theory, $E(\varepsilon_{it}) = 0$ and $E(\varepsilon^2_{it}) = \sigma^2_{\varepsilon_i}$, random variable ε with the mean of null and variance of $\sigma^2_{\varepsilon_i}$ is standardized as it follow where σ_ε is the standard variance of ε and X is the random variable with the mean of W and variance of σ^2.

$$\boldsymbol{\varepsilon_{it} = \sigma_{\varepsilon_i} \frac{X - \omega}{\delta}} \tag{2}$$

Assume that random variable of x has asymmetric distribution in CAPM-IAPD and asymmetric exponential distribution in AEPD. Then we can describe it as it follows:

$$X = \omega + \frac{\delta}{\sigma_\varepsilon} \varepsilon = \omega + \frac{\delta}{\sigma_\varepsilon} (Z_{it} - \alpha_{Mi} - \beta_{Mi} Z_M) \tag{3}$$

In which Z and Z_m were respectively determined as the random variable of excess return and Z_m as the non-random variable. To determine Asymmetric Power Distribution for return variable Z is used as :

$$f_Z(z) = f_X(x) \left| \frac{d_Z}{d_X} \right|^{-1} = f_X\left(\omega + \delta . \frac{Z - \alpha_M - \beta_M Z_M}{\sigma_\varepsilon}\right) . \frac{\delta}{\sigma_\varepsilon} \tag{4}$$

And Data Generation Process in CAPM-IAPD would be as:

$$Z_{it} = \alpha_{iM} + \beta_{iM} Z_{Mt} + \varepsilon_{it}, \qquad \varepsilon_{it} \sim \mathrm{IAPD}(\alpha, \lambda, 0, \sigma^2_{\varepsilon_i}),$$

$$\varepsilon_{it} = \sigma_{\varepsilon_i} \frac{X - \omega}{\delta}$$

$$E(\varepsilon_{it}) = 0 \tag{5}$$

$$E(\varepsilon^2_{it}) = \sigma^2_{\varepsilon_i}$$

The next distribution which use in this paper is exponentially asymmetric independent distribution.Considering exponential distribution function and its fat tails, DGP in CAPM-IAEPD is as follows:

$$Z_{it} = \alpha_{iM} + \beta_{iM} Z_{Mt} + \varepsilon_{it} \qquad \varepsilon_{it} \sim \text{IAEPD}\left(\alpha, P_1, P_2, 0, \sigma^2_{\varepsilon_i}\right)$$

$$\varepsilon_{it} = \sigma_{\varepsilon_{it}} \frac{Y-\omega}{\delta}$$

$$\text{E}\ (\varepsilon_{\text{it}}) = 0 \tag{6}$$

$$\text{E}\ (\varepsilon^2_{it}) = \sigma^2_{\varepsilon_{it}}$$

These models help us estimate returns and risk in situations where the distribution of assets is not normal.[13]

2.3 Entropy

After estimating the return and risk using the mentioned methods, we need to optimize and select the optimal portfolio and analyze it with appropriate methods. To do so, in addition to higher moments of the model, we use Shannon and Gini-Sipmson entropy to optimize our model.

Measuring the amount of calculated uncertainty in Shannon entropy uses the following equation:

$$E_i = S(p_1, p_2, p_3, \dots, p_n) = -k(\textstyle\sum_{i=1}^{n} p_i \times Lnp_i) \tag{7}$$

Here K is a fixed amount and is used to make sure that E_i is going to be between null and one and is calculated as:

$$\text{K}= \frac{1}{Ln(m)}$$

The main index of Simpson λ is equal to the probability that two values randomly chosen from a set of data that are under evaluation (with substitution) are the same type. Transformation of 1- λ is equal to the probability that two values show different types. Also in ecology, this measurement is known as the probability of special impact and Gini-Simpson index which is accounted for as a revolution in regular diversification:

$$1\text{-}\lambda = 1\text{-}\textstyle\sum_{i=1}^{R} p_i^2 = 1\text{-}\frac{1}{D^2} \tag{8}$$

In matrix form is equal to:

$$E_{G-S} = 1 - \textstyle\sum_{i=1}^{n} w_i^2 = 1 - W^T W \tag{9}$$

Therefore in this research in addition to optimization of higher moments, Shannon and Gini-Simpson entropies are used to improve the optimization of the portfolio.[22]

In the second phase of the research, to find the optimum portfolio, we will use higher moments optimization and then we will analyze the effect of different weights on adjusted Sharpe ratio value in the third moment. With this, we will chose the best portfolio from the aspects of return and risk.

3 Research Methodology

This is an empirical study in the descriptive research category. The population in this research includes all the companies registered in Tehran Stock Exchange (TEDPIX). Sampling was conducted by 217 companies and in the period of March 20th 2011 to the end of February 2017. After summarization, the data was categorized in 11 groups. Financial information related to stock index were gathered from Rahavard Novin software and Bourseview.com.

Statistical steps are as it follows:

1. weekly return of the sample companies are gathered and stock index and risk-free index information in each month are extracted from Rahavard Novin software and Boursevies.com.
2. Using SPSS software, we analyze whether the return for 11 investment group is normal or not.
3. Utilizing R software, generalized capital asset pricing models – with taking the aforementioned assumption into account – are coded and both return and risk of the assets that are being studied are calculated and informative criteria of Akaike and Schwarz are assessed.
4. Due to limitations, we optimize higher moments using MATLAB software. By SPSS, a statistical test is conducted on the data and hypotheses are tested to be accepted or rejected .
5. Finally,last step utilizing the chosen weights, the effect of each moment on the optimized portfolio is observed and they were compared by adjusted Sharpe ratio.

3.1 Model Estimation

In the following, we will estimate the return and risk in two models, CAPM-IAPD and CAPM-IAEPD**.** ε_{it} is IAPD distribution error with the mean of null and variance of $\sigma^2_{\varepsilon_i}$. Mean and vaiance of x are:

$$\omega = \mathrm{E(X)} = \frac{\Gamma(2/\lambda)}{\Gamma(1/\lambda)}(1-2\mathrm{a})\delta_{\alpha,\lambda}^{-1/\lambda}$$

$$\delta^2 = Var(X) = \frac{\Gamma(3/\lambda)\Gamma(1/\lambda)(1-3\alpha+3\alpha^2)-\Gamma(2/\lambda)^2(1-2\alpha)^2}{\Gamma(1/\lambda)^2}\delta_{\alpha,\lambda}^{-2/\lambda} \qquad (10)$$

in which α is the skewness parameter and measures asymmetry degree in range of (0,1) and λ>0 is distribution sequence variable as:

$$\delta_{\alpha,\lambda}= \frac{2\alpha^\lambda(1-\alpha)^\lambda}{\alpha^\lambda+(1-\alpha)^\lambda} \qquad \delta_{\alpha,\lambda} \epsilon\ (\cdot,1) \qquad (11)$$

and $\Gamma\,(\cdot)$ is Gama distribution variable.

So,density function of x is:

$$f_x(x) = \begin{cases} \left(\frac{\delta_{\alpha,\lambda}^{1/\lambda}}{\Gamma(1+1/\lambda)}\right)\exp\left(-\frac{\delta_{\alpha,\lambda}}{\alpha^\lambda}|x|^\lambda\right), & \text{for } x\ \leq 0 \\ \left(\frac{\delta_{\alpha,\lambda}^{1/\lambda}}{\Gamma(1+1/\lambda)}\right)\exp\left(-\frac{\delta_{\alpha,\lambda}}{(1-\alpha)^\lambda}|x|^\lambda\right), & \text{for } x > 0. \end{cases} \qquad (12)$$

In Capital Asset Pricing Model with the assumption of an independent and uniform AEPD (CAPM-IAEPD) mean and variance are:

$$\omega = E(Y) = \frac{1}{B}\left[(1-\alpha)^2\frac{P_2\Gamma(2/P_2)}{\Gamma^2(1/P_2)} - \alpha^2\frac{P_1\Gamma(2/P_1)}{\Gamma^2(1/P_1)}\right] \qquad (13)$$

$$\delta^2 = \mathrm{Var(Y)}=\frac{1}{\mathrm{B}^2}\{(1-\alpha)^3\frac{\mathrm{P}_2^2\Gamma(2/\mathrm{P}_2)}{\Gamma^3(1/\mathrm{P}_2)} + \alpha^3\frac{\mathrm{P}_1^2\Gamma(3/\mathrm{P}_1)}{\Gamma^3(1/\mathrm{P}_2)} - \left[(1-\alpha)^2\frac{\mathrm{P}_2\Gamma(2/\mathrm{P}_2)}{\Gamma^2(1/\mathrm{P}_2)} - \alpha^2\frac{\mathrm{P}_1\Gamma(2/\mathrm{P}_1)}{\Gamma^2(1/\mathrm{P}_1)}\right]$$

and p_1 and p_2 are respectively left and right sequences. Standard density function of AEPD is defined as:

$$f_Y(y)=\begin{cases}\left(\frac{\alpha}{\alpha^*}\right)K_{EP}(P_1)\exp\left(-\frac{1}{P_1}\left|\frac{y}{2\alpha^*}\right|^{P_1}\right) & \text{for } y\le 0\\ \left(\frac{1-\alpha}{1-\alpha^*}\right)K_{EP}(P_2)\exp\left(-\frac{1}{P_2}\left|\frac{y}{2(1-\alpha^*)}\right|^{P_2}\right) & \text{for } y>0.\end{cases} \qquad (14)$$

3.2 Higher Moment Optimization

considering that risk and return variables which were calculated by previous capital asset pricing models, now is the time foroptimizing higher moments. To do so, first we introduce the required variables as it follows:

In this segment, by utilizing Polynominal Goal Programing (PGP), we optimize higher moment. Assume that the transpose matrix of asset weight is W^T =(w_1 و w_2 و ... و w_n), in which W_i is the weight of i^{th} asset with risk in the portfolio. Furthermore, the value of R and M=$(m_1$ و m_2 و ... و $m_n)^T$ is used as distribution and assets return matrix. The values of V,S and K are respectively matrixes of variance-covariance, skewness and kurtosis.

$$R_p = E(R_p) = W^T M = \sum_{i=1}^{n} w_i m_i \qquad (15)$$

$$V_p = V(R_p) = W^T V(W) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \qquad (16)$$

$$S_p = S(R_p) = E(W^T(R-M))^3 = \sum_{i=1}^{n}\sum_{j=2}^{n}\sum_{k=1}^{n} w_i w_j w_k s_{ijk} \qquad (17)$$

$$K_p = K(R_p) = E(W^T(R-M))^4 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} w_i w_j w_k w_l k_{ijkl} \qquad (18)$$

in which S_{ijk} and k_{ijkl} are defined as it follows:

$$s_{ijk} = E[(R_i - m_i)(R_j - m_j)(R_k - m_k)] \qquad (19)$$

$$k_{ijkl} = E[(R_i - m_i)(R_j - m_j)(R_k - m_k)(R_l - m_l)] \qquad (20)$$

The relative values of skewness and kurtosis are Ku(R_p)=$\frac{S(R_p)}{\sigma_p^4(R_p)}$ and SK(R_p)=$\frac{S(R_p)}{\sigma_p^3(R_p)}$ and the values of K and S as Shannon-Simpson entropy index values are defined as:

$$E_s = -\sum_{i=1}^{n} w_i \ln w_i = -W^T \ln W \qquad (21)$$

$$E_{G-S} = 1 - \sum_{i=1}^{n} w_i^2 = 1 - W^T W \qquad (22)$$

where E_s and E_{G-S} are portfolios.

The objective function of first to fourth moments model and Shannon-Simpson entropy is:

$$P(1)=\begin{cases} Maximize & W^T M\\ \text{Minimize} & W^T V(W)\\ Maximize & E(W^T(R-M))^3\\ \text{Minimize} & E(W^T(R-M))^4\\ Maximize & -W^T(\ln W)\\ Maximize & 1-W^T W\\ \text{Subject to} & W^T 1_N = 1\\ & W \ge 0\end{cases} \qquad (23)$$

there are two steps in Polynominal Goal Programing. First, focusing on each objective function and the optimized calculation without considering other objective functions which are shown as ،R_{pe}^{*} K_{p}^{*}،S_{p}^{*}،،E_{s}^{*} V_{p}^{*} E_{G-S}^{*}.

Second, the goal values of d_2،d_3،d_4،d_4،d_5،d_6 d_1، are used to minimize variance from expected levels. The expected levels are produced by a one by one calculation of 6 sub-function.

$$SP(1) = \begin{cases} \text{Maximize} \quad R_{pe}^{*} = W^{T}M \\ \text{subject to} \quad W^{T}1_{N} = 1 \\ \quad W \geq 0 \end{cases} \tag{24}$$

$$SP(2) = \begin{cases} \text{Minimize} \quad V_{p}^{*} = W^{T}V(W) \\ \text{subject to} \quad W^{T}1_{N} = 1 \\ \quad W \geq 0 \end{cases} \tag{25}$$

$$SP(3) = \begin{cases} \text{Maximize} \quad S_{p}^{*} = E(W^{T}(R-M))^{3} \\ \text{subject to} \quad W^{T}1_{N} = 1 \\ \quad W \geq 0 \end{cases} \tag{26}$$

$$SP(4) = \begin{cases} \text{Minimize} \quad K_{p}^{*} = E(W^{T}(R-M))^{4} \\ \text{subject to} \quad W^{T}1_{N} = 1 \\ \quad W \geq 0 \end{cases} \tag{27}$$

$$SP(5) = \begin{cases} \text{Maximize} \quad E_{s}^{*} = -W^{T}(lnW) \\ \text{subject to} \quad W^{T}1_{N} = 1 \\ \quad W \geq 0 \end{cases} \tag{28}$$

$$SP(6) = \begin{cases} \text{Maximize} \quad E_{G-S}^{*} = 1 - W^{T}W \\ \text{subject to} \quad W^{T}1_{N} = 1 \\ \quad W \geq 0 \end{cases} \tag{29}$$

The above mentioned models can be solved by using a linear or non-linear programming. Now, these objective functions can be accumulated using Minowski Distance in PGP model. Minowski distance is defined as it follows:

$$Z=(\sum_{k=1}^{m} \left|\frac{d_k}{Z_k}\right|^{p})^{1/p} \tag{30}$$

In the above mentioned equation Z_k shows the standardized values of K^{th} objective function and d_k shows the variance from K^{th} objective function. Furthermore, investors have their own priorities in among different goals which priorities of each objective function is shown by λ_i. Considering different values for λ_i can change the optimization model.

In the next part, by utilizing equations of expected levels simultaneously and knowing the value of λ_i, the optimum value of each objective function in the second step of optimization model would be changed to:

$$\begin{cases} Minimize\ Z = \left(1+\left|\frac{d_1}{R_{pe}^*}\right|\right)^{\lambda_1} + \left(1+\left|\frac{d_2}{V_p^*}\right|\right)^{\lambda_2} + \left(1+\left|\frac{d_3}{S_p^*}\right|\right)^{\lambda_3} + \left(1+\left|\frac{d_4}{K_p^*}\right|\right)^{\lambda_4} + \left(1+\left|\frac{d_5}{E_s^*}\right|\right)^{\lambda_5} + \left(1+\left|\frac{d_6}{E_{G-S}^*}\right|\right)^{\lambda_6} \\ \quad subject\ to : W^T M + d_1 = R_{pe}^* \\ \quad W^T M - d_2 = V_p^* \\ \quad E(W^T(R-M))^3 + d_3 = S_p^* \\ \quad E(W^T(R-M))^4 - d_4 = K_p^* \\ \quad -W^T(lnW) + d_5 = E_s^* \\ \quad 1 - W^T W + d_6 = E_{G-S}^* \\ \quad W^T 1_N = 1 \\ \quad w \geq 0 \\ \quad d \geq 0 \end{cases} \tag{31}$$

In the last step, the produced portfolios will be compared using performance ratio criteria and then, the best one will be chosen. For this purpose, we will use Iralsen-Adjusted-Sharpe ratio and adjusted skewness criteria.

3.3 Performance Index

Sharp criterion is one of the most widely used criteria for evaluating portfolio performance.To calculate this criterion:

$$SR=\frac{r_p^0}{\sigma_p^0} \tag{32}$$

In the above mentioned equation $\boldsymbol{r_p^0}$ and $\boldsymbol{\sigma_p^0}$ are respectively mean and variance. SR index is a proper criterion to measure portfolio performance when $\boldsymbol{r_p^0}$ has a negative value. In this index we must use Iralsen-Adjusted-ratio index in which abs show the absolute values:

$$MSR = \frac{r_p^0}{\sigma_p^{\,r_p^0 / abs(r_p^0)}} \tag{33}$$

Since Sharpe ratio is based on mean and variance theory, it is only valid when the data has normal distribution. Therefore, Sharpe criterion can have a misleading answer to distributions with skewness and fat tails. To solve this problem, we use adjusted ratio with skewness:

$$ASR = \mathrm{SR} \times \sqrt{1 + \frac{SK(R_p) \times SR}{3}} \tag{34}$$

4 Emperical Results

4.1 Normality Test

In this research, in the first step, we will examine whether assets return is normal or not. To do so, we use Kologorov-Smirnov test in which we had these results:

Table 1: The Result of Kolmogorov-Smirnov Test

Results	Degree of freedom	Significance level	statistic	Group	

Reject or null	7126	0.00	0.128	Concrete sector	1
Reject or null	1172	0.00	0.130	Medicine sector	2
Reject or null	2912	0.00	0.095	Metal sector	3
Reject or null	7196	0.00	0.139	Sugar sector	4
Reject or null	398	0.00	0.115	Mineral sector	5
Reject or null	557	0.00	0.133	Ceramic sector	6
Reject or null	1213	0.00	0.151	Automotive sector	7
Reject or null	809	0.00	0.119	Construction sector	8
Reject or null	5147	0.00	0.119	Petrochemical sector	9
Reject or null	2788	0.00	0.098	Investment sector	10
Reject or null	2842	0.00	0.114	Other sectors	11

By considering Kolmogorov-Smirnov test's statistic in different groups and the reliability level that we are executing our study, the meaningfulness level is less than 0.05 and the null hypothesis declaring that the data are normal is rejected. In this regard, return distribution of under-study groups are not normal ones.

4.2 CAPM (IIAPD) and CAPM (IAEPD) Return and Variance Estimation

To estimate return and risk in each CAPM we use R software. Table 2 illustrates mean of return and risk of 30 chosen sample in each model.

Table 2: The result of Average Return and Risk of 30 Selected Samples

CAPM-IAEPD		CAPM-IIAPD		
Variance	**Return**	**Variance**	**Return**	
0.0448	0.0251	0.0563	0.0199	1
0.0493	0.0608	0.0375	0.0178	2
0.0556	0.0058	0.0336	0.0173	3
0.0484	0.0256	0.0386	0.0241	4
0.0443	0.0696	0.0388	0.0435	5
0.0538	0.0040	0.0358	0.0425	6
0.0582	0.0051	0.0312	0.0130	7
0.0632	0.0467	0.0344	0.0136	8
0.0479	0.0392	0.0378	0.0261	9
0.0445	0.0656	0.0336	0.0364	10
0.0489	0.0491	0.0398	0.0220	11
0.0540	-0.0049	0.0287	0.0320	12
0.0483	0.0572	0.0400	0.0244	13
0.0441	0.0602	0.0315	0.0297	14

0.0442	0.0138	0.0389	0.0239	15
0.0467	0.0685	0.0358	0.0287	16
0.0481	-0.0580	0.0403	0.0250	17
0.0487	0.0321	0.0368	0.0297	18
0.0539	-0.0704	0.0010	0.000	19
0.0481	-0.0518	0.0354	0.0219	20
0.0504	-0.0757	0.0409	0.0175	21
0.0487	0.0206	0.0396	0.0251	22
0.0472	-0.0617	0.0359	0.0178	23
0.0494	0.0511	0.0360	0.0149	24
0.0480	-0.0363	0.0395	0.0258	25
0.0378	-0.0262	0.0450	0.0351	26
0.0388	0.0012	0.0388	0.0277	27
0.0489	0.0543	0.0392	0.0204	28
0.0472	0.0568	0.0400	0.0303	29
0.0470	-0.0513	0.0383	0.0290	.30

4.3 Goodness of Fit

Following this, we will search for the best fitting model for the data. For this reason, we will use Akaike and Schwartz information criterion and doing so we will categorize the best models. Here, the lesser the value of Schwartz criterion is, the better the chosen model is, compared to the other ones. Results in table-3 demonstrates the best model.

Table3 : The Result of Information Criterion for Fitted Models

Model	Akaike criterion(AIC)	Schwarts criterion (BIC)
CAPM-IAPD	-250.70	-238.67
CAPM-IAEPD	-246.70	-232.130

The result for Akaike and Schwartz statistic:

CAPM with independent and identically asymmetric power distributed (CAPM-IIAPD), is better than CAPM with asymmetric independent and identically asymmetric exponential power distributed with two tail parameters (CAPM-IAEPD). Also, Schwartz statistic of CAPM-IIAPD is better than CAPM-IAEPD.

As a result, considering the results for Akaike and Schwartz statistics of CAPM-IIAPD has a better approach in comparison to CAPM-IAEPD.

4.4 Model Review Using Different Moments

Table 4 shows objective functions of (CAPM-IIAPD):

Table 4 :The Results For CAPM-IIAPD Objective Functions

Simpson entropy mean	Shannon entropy mean	Kurtosis mean (×10-4)	Skewness mean (×10-4)	Variance mean	Return mean	Model	
1.348	7.5258	0.0362	-0.0272	0.0336	0.0173	EWM	1
0.7797	1.5548	0.0011	-0.0022	0.0101	0.0223	MVM	2
0.7787	1.555	9.4843	-0.0020	0.0101	0.0224	MVSM	3
0.7089	1.3069	5.2919	-1.8704	0.014	0.0256	MVSKM	4
0.7137	1.3327	5.4504	-1.9474	0.0138	0.0252	MVSKE_SM	5
0.7170	1.3418	5.6005	-1.9919	0.0135	0.0249	MVSK E_{G-S} M	6

As it is shown in table 4, the highest values in return means among MVSM, MVM, EWM and MVSKM models belongs to MVSKM model. When we consider entropy models too, MVSKE$_{G\text{-}S}$ and MVSKE$_S$M have equal efficiencies. The best variance (V_p) belongs to MVM model. The most optimum third moment (S_p) belongs to MVSM model. The best Shannon entropy (E_s) belongs to EWM and the best Gini-Simpson entropy ($E_{G\text{-}S}$) belongs to EWM.

Now if MVM, MVSM and MVSKM are compared to one another, MVSKM will have the best R_p and K_p, and MVM and MVSM will have the best variance value (V_p). MVSM has the best third moment value (S_p) but EWM has the best Shannon entropy (E_s) and mean Gini-Simpson entropy ($E_{G\text{-}S}$) values.

In addition, when we compare MVSKE$_S$M and MVSKE$_{G\text{-}S}$ models, we can conclude that MVSKE$_{G\text{-}S}$ has the best variance (V_p), Shannon entropy (E_S) and Gini-Simpson entropy ($E_{G\text{-}S}$) values. On the other hand, MVSKE$_S$M has more optimum return (R_p) and Kurtosis (K_p) values when compared to MVSKE$_{G\text{-}S}$.

And the second CAPM model is CAPM-IAEPD which is illustrated in table 5.

Table 5 : The Results of CAPM-IAEPD Objective Functions

Simpson entropy mean	Shannon entropy mean	Kurtosis mean (×10-4)	Skewness mean (×10-4)	Variance mean	Return mean	Model	
2.0485	1.9051	0.0362	-0.0272	0.0556	0.0058	EWM	1
0.5955	0.9959	0.0046	-0.0066	0.0219	0.0146	MVM	2
0.5944	0.9934	0.0043	-0.0063	0.0219	0.0145	MVSM	3
0.5762	1.1449	7.5674	-0.00297	0.0194	0.028	MVSKM	4
0.6012	1.1922	7.1460	-0.00271	0.0183	0.0291	MVSKE_SM	5
0.6229	1.2305	7.8745	-0.0026	0.0174	0.0302	MVSK E_{G-S} M	6

As it is demonstrated in table 5, the highest values in return means among MVSM, MVM, EWM and MVSKM models belongs to MVSKM model. When we consider entropy models too, MVSKE$_{G\text{-}S}$ has a higher return. From the aspect of fluctuations, the best variance (V_p) belongs to MVSKE$_{G\text{-}S}$. The best

Shannon entropy and Gini-Simpson entropy belongs to $MVSKE_SM$. Also, the best fourth moment belongs to MVM.

Now if MVM, MVSM and MVSKM are compared to one another, MVSKM will have the best R_p and V_p, and MVM and MVSM will have the best third moment (S_p). MVSKM has the best skewness (K_p) and in MVSKM the best entropy E_s, Gini-Simpson entropy mean values are approximately equal in all three models.

In addition, when we compare $MVSKE_SM$ and $MVSKE_{G\text{-}S}$ models, we can conclude that $MVSKE_{G\text{-}S}$ has the best variance (V_p), return (R_p), third moment (S_p), Shannon entropy (E_S) and Gini-Simpson entropy ($E_{G\text{-}S}$) values.

4.5 Testing the Research Hypothesis

4.5.1 First Hypothesis

By taking the data acquired from higher moment optimization for CAPM-IIAPD and CAPM-IAEPD into account, the following results were found.

Table 6 : The Results For Comparison of Adjusted Return Mean Approach in CAPM-IIAPD and CAPM-IAEPD for Third Moments

	results	**CAPM-IAEPD**		**CAPM-IIAPD**		
Significance level	**statistic t**	**Risk**	**Return**	**Risk**	**Return**	
0.9552	0.2419	0.022	0.015	0.01	0.022	**MVSM**

According to the results reported in table-6, we can accept the hypothesis of equal adjusted return based on optimized portfolio risk based on MVSM using approach CAPM-IAEPD compared to portfolio with approach CAPM-IIAPD. In other words, there is no meaningful difference in approach of CAPM-IIAPD and CAPM-IAEPD.

4.5.2 Second Hypothesis

Table7: The Results For Comparison of Adjusted Return Mean Approach in CAPM-IIAPD and CAPM-IAEPD for Third and Fourth Moments

	results	**CAPM-IAEPD**		**CAPM-IIAPD**		
Significance level	**statistic t**	**Risk**	**Return**	**Risk**	**Return**	
0.0387	2.9215	0.019	0.028	0.014	0.026	**MVSKM**

According to the reported results in table 7, we can reject the hypothesis of equal adjusted return based on optimized portfolio risk based on MVSM using approach CAPM-IAEPD compared to portfolio with approach CAPM-IIAPD. In other words, there is no meaningful difference in approach of CAPM-IIAPD and CAPM-IAEPD in second to fourth moments (the statistic value of the test is less than t-student value - $t_{58,95\%} = 2.002$ – and the meaningfulness level is less than 0.05.)

4.6 Performance Index

Typically, optimization is done with binary values and the effect of each moment and chosen entropies, does not have different weights for λ_i. In this regard, in order to determine the effect of each constraint function of the model, we will consider all the possibilities (For example, in the first instance the values $\lambda_1 = 3$ $\lambda_2 = 1$، ، $\lambda_3 = 1$، ، $\lambda_4 = 1$ و $\lambda_5 = 1$ $\lambda_6 = 0$ is shown as (3-1-1-1-1-0).) We pay attention to the point that the highest value is $\lambda_\mathrm{i} = 3$ and the least value is $\lambda_i = 0$. When $\lambda_i = 0$ it means that corresponding objective function is not consider. Taking higher moments values into account, performance analysis criteria in two levels for criteria Iralsen adjusted Sharpe and adjusted Sharpe by skewness is as it follows:

Table 8: Comparison of The Results For The Models (With the Assumption of Unequal weights) Using Performance Analysis Criteria

Model						Priority
CAPM-IAEPD		CAPM-IIAPD		CAPM		
ASR	MSR	ASR	MSR	ASR	MSR	
0.1063	0.1036	0.1333	0.1440	0.0637	0.1123	0-1-1-1-1-3
0.1061	0.1061	0.1293	0.1407	0.0630	0.1118	1-0-1-1-1-3
0.1079	0.1079	0.1501	0.1638	0.0601	0.1131	0-3-1-1-1-3
0.1075	0.1075	0.1490	0.1625	0.0608	0.1131	3-0-1-1-1-3
0.1057	0.1057	0.1234	0.1328	0.0635	0.1123	0-0-1-1-1-3
0.2577	0.3005	0.2167	0.2413	0.0591	0.1010	0-1-1-1-3-1
0.2587	0.3013	0.2167	0.2413	0.0597	0.1019	1-0-1-1-3-1
0.2592	0.3064	0.2131	0.2395	0.0535	0.0978	0-3-1-1-3-1
0.2636	0.3101	0.2164	0.2414	0.0583	0.1023	3-0-1-1-3-1
0.2571	0.2986	0.2183	0.2422	0.0602	0.1015	0-0-1-1-3-1
0.2197	0.2490	0.2164	0.2380	0.0642	0.0897	0-3-1-1-1-1
0.2252	0.2554	0.2055	0.2228	0.0746	0.0944	3-0-1-1-1-1

The comparison of performance analysis results has some interesting points. Considering two performance analysis criteria of MSR and ASR, the following results were produced:

In traditional CAPM model the highest amount of MSR was belong to (3-1-1-1-3-0) model. Since in the optimization process higher moments were also used, it is better to use more efficient and up-to-date methods in performance analysis. Therefore, the highest amount of ASR belongs to (3-0-1-1-1-1) model. In other words, with the addition of entropy (Gini-Simpson) to the model, portfolio performance was improved.

In CAPM-IIAPD model, the highest MSR amount belongs to (3-1-1-1-0-0) model. Since in the optimization process higher moments were also used, it is better to use more efficient and up-to-date methods in performance analysis. Therefore, the highest amount of ASR belongs to (1-3-1-1-0-0) model. The advantage of this criterion over MSR is the consideration of third moment (Skewness) in performance analysis. In this model, entropy has no effect on increasing the performance of the model and the best model is for the highest weight on second moment.

In CAPM-IAEPD model, the highest MSR amount belongs to (1-3-1-1-0-3) model. Since in the optimization process higher moments were also used, it is better to use more efficient and up-to-date methods in performance analysis. Therefore, the highest amount of ASR belongs to (1-3-1-1-0-3) model. The advantage of this criterion over MSR is the consideration of third moment (Skewness) in performance analysis. Therefore, with the addition of Gini-Simpson entropy model's performance will be improved.

5 Conclusions and Implications

In this research we we set different goals. First, we wanted to present a model that when markets face financial crises and distribution of the asset return no longer is a normal one, we would be able to use this distribution so that there would be minimum deviation from real data. To do so, we used weekly return data of companies registered in Iran's investment market. After that, with the help of Kolmogorov-Smirnov test, we rejected the hypothesis of normal return for the companies under study. Following that, we computed CAPM-IIAPD and CAPM-IAEPD models. Then, using Akaike and Schwartz statistics, we categorized the models and as a result found that CAPM-IIAPD model is superior to CAPM-IAEPD. Therefore, CAPM-IIAPD and CAPM-IAEPD are more desirable performance in fitting real financial data to estimate return and risk than Traditional CAPM.

After choosing the best models for return and risk estimation as entry variables for portfolio optimization, optimization was conducted via Polynominal Goal Programming. To determine the effect of each moment and entropy's effect, different values were given to λ_i and in CAPM the highest return was belong to (3-1-1-1-0-0) model with highest weight approach first moment and (1-3-1-1-1-0) model with highest weight approach second moment had the least variance. These values in CAPM=IIAPD were also considered and the highest return belonged to (1-3-1-1-0-0) with the highest weight approach variance and (1-1-1-1-3-0) model with the least variance that this model emphasizes on the maximum weight of Shannon entropy. Finally, in CAPM-IAEPD the highest return belonged to (3-1-1-1-0-3) model with maximum weights for second moment and Gini-Simpson entropy and (1-3-1-1-0-3) model with the highest weight approach variance, has the lease variance.Therefore, According to different weights, portfolio optimization with different moment was estimated and the effect of each moment was investigated.

After optimization via higher moments, it is time for hypothesis test. The third moment test shows that there is not meaningful difference between two approach of CAPM-IIAPD and CAPM-IAEPD. On the other hand, the hypothesis of equal adjusted return for optimized portfolio risk based on MVSKM model via CAPM-IAEPD approach in relevance to CAPM-IIAPD approach was rejected and there is a meaningful difference between two pricing approach when optimizing second and fourth moments.

Since the asset return distribution does not follow normal distribution,we use the adjusted Sharp ratios to evaluate the performance so that the third and fourth moments in the performance are also examined. Finally, we calculate ASR and MSR ratios to evaluate the performance of the optimized portfolios. In CAPM-IIAPD the highest ASR value was belong to (1-3-1-1-0-0) model and entropy did not have any effect on increasing the performance level of the model and the best model was with the highest weight on second moment. In CAPM-IEAPD the highest ASR value was belong to (1-3-1-1-0-3) and with the addition of Gini-Simpson entropy, the performance would increase.

In this research, we developed a proper pricing model for abnormal efficiencies and financial crisis and using higher moments, we issued the problem in solving portfolio optimization in financial crises, drastic fluctuations and etc. and finally for the times that we are using higher moments, we utilized a more proper performance analysis criterion compared to Sharpe criterion to compare optimized portfolio .

6 Suggestions

In the following, we will provide practical suggestions and suggestions for future research:

- In asymmetric capital asset pricing models, an undesirable risk criterion can be used instead of an absolute risk criterion (variance) to calculate risk.
- As a practical suggestion, investment managers can use the asymmetric capital asset pricing models to predict returns and risk and use higher torque optimization to allocate the assets under their management in the market and decision capital. To.
- It is recommended that problem solving methods be developed and meta-heuristic algorithms be used.
- In addition to Shannon and Ginny-Simpson entropies, it is recommended to use other entropy functions to optimize the portfolio.
- In addition, the models are checked using fuzzy logic to check the performance of the models.

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