



# Numerical Solution of Multidimensional Exponential Levy Equation by Block Pulse Function

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## ABSTRACT

The multidimensional exponential Levy equations are used to describe many stochastic phenomena such as market fluctuations. Unfortunately in practice an exact solution does not exist for these equations. This motivates us to propose a numerical solution for  $n$ -dimensional exponential Levy equations by block pulse functions. We compute the jump integral of each block pulse function and present a Poisson operational matrix. Then we reduce our equation to a linear lower triangular system by constant, Wiener and Poisson operational matrices. Finally using the forward substitution method, we obtain an approximate answer with the convergence rate of  $O(h)$ . Moreover, we illustrate the accuracy of the proposed method with a 95% confidence interval by some numerical examples.

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## 1 Introduction

The multidimensional exponential Levy equation is a powerful tool for modelling of market fluctuations, both for hedging and option pricing goals. There are a considerable volume of research articles related to this topic in different financial mathematics journals; see [1-6]. Unfortunately, these stochastic differential equations have not an exact solution and we cannot solve explicitly them. So we have to approximate the answer by numerical methods. In this regard, during recent years various tools such as Block Pulse Functions (BPFs), Fourier series, Chebyshev polynomials, etc., were applied to find an approximate solution for such stochastic systems, see [7-20]. Orthogonal functions block pulse has been used for solving many problems. Specially, in stochastic systems Maleknejad et al. [10] applied BPF for an  $m$ -dimensional linear stochastic Itô–Volterra integral equation and presented the approximate solution with the convergence rate  $O(h)$ . Also Maleknejad et al. [11] estimated numerical solutions of linear stochastic Volterra integral equations by BPF and stochastic operational matrix. Khodabin et al. [12] employed BPF for estimating an approximate answer for linear stochastic Volterra–Fredholm integral equation. In all of aforementioned papers authors reduced stochastic equations to the triangle system by introducing stochastic operational matrices. In current text we extend previous works and employ BPF for multidimensional exponential Levy equations. We propose a Poisson operational matrix for the jump integral of each BPF and convert our problem to a linear lower triangular system by operational matrices and then solve it by the forward substitution method. The advantage of BPF approach compared to other methods is that due to some properties such as orthogonality, disjointness, piecewise constant trajectories, etc., calculations are simple and effective. We consider the complete filtered probability space  $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, P)$ , where  $P$  is the probability measure and  $\{\mathcal{F}_t\}_{t \geq 0}$  is the filtration produced by the stock price process  $\{S_t\}_{t \geq 0}$ . We suppose

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that the dynamic of the stock price process is modeled by the multidimensional exponential Levy equation, as follows:

$$S(t) = r(t) + \int_0^t S(u^-)\mu(u,t)du + \sum_{i=1}^p \int_0^t S(u^-)\sigma_i(u,t)dW_i(u) + \sum_{j=1}^q \int_0^t \int_{\mathbb{R}^+} S(u^-)\eta(z)\zeta_j(u,t)\tilde{N}_j(du, dz) \tag{1}$$

where

- $W(t)$  is the Wiener process.
- $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$  is the compensated Poisson random measure with intensity measure  $\nu(\cdot)$  on  $\mathbb{R}^+$ .
- Components of  $p$ -dimensional Wiener process  $W = (W_1, \dots, W_p)$  and  $q$ -dimensional Poisson process  $N = (N_1, \dots, N_q)$  are mutually independent as well as we set  $p = q = n$ .
- $\mu(u,t)$ ,  $\sigma_i(u,t)$  and  $\zeta_j(u,t), i, j = 1, \dots, n$ , are measurable functions.

The current study proceeds as follows: In section 2, we summarize some basic features of BPFs. In Section 3, we review the operational matrix [8] and the stochastic one [10]. Also, we calculate the jump integral of each BPF and present the Poisson operational matrix. In Section 4, we simplify our equation to a triangle system by operational matrices and then we solve it by the forward substitution method. In section 5, we show that the convergence rate of the proposed method is  $O(h)$ . Finally, we provide some numerical examples in section 6 and verify the accuracy of our approach with a 95% confidence interval.

## 2 Block Pulse Functions (BPFs)

In this section we recall some key definitions and features of BPFs, for more details see [8, 9]. Using these relations we can simplify computations and solve our problem.

**Definition 1.** A block pulse function  $\psi_i(t), i = 1, 2, \dots, n$ , is defined by

$$\psi_i(t) = \begin{cases} 1 & (i-1)h \leq t < ih \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where  $h = \frac{T}{n}$  in the interval  $[0, T)$ .

From above definition, we derive BPFs are disjoint, i.e.,

$$\psi_i(t)\psi_j(t) = \begin{cases} \psi_i(t) & i = j \\ 0 & i \neq j, \end{cases} \tag{3}$$

orthogonal with each other in  $[0, T]$ , i.e.,

$$\int_0^T \psi_i(t)\psi_j(t)dt = \begin{cases} h & i = j \\ 0 & i \neq j, \end{cases} \tag{4}$$

and for an arbitrary function  $g \in L^2([0, T])$ , as  $n \rightarrow \infty$ , the Parseval's identity is satisfied, i.e.,

$$\int_0^T g^2(t)dt = \sum_{i=1}^{\infty} g_i^2 \|\psi_i(t)\|^2,$$

Where

$$g_i = \frac{1}{h} \int_0^T g(t) \psi_i(t) dt, \quad (5)$$

for  $i, j = 1, 2, \dots, n$ .

Then we can write

$$\Psi(t) \Psi^T(t) = \begin{pmatrix} \psi_1(t) & 0 & \dots & 0 \\ 0 & \psi_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \psi_n(t) \end{pmatrix}_{n \times n}, \quad (6)$$

$$\Psi^T(t) \Psi(t) = 1,$$

and

$$\Psi(t) \Psi^T(t) G^T = \begin{pmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_n \end{pmatrix}_{n \times n} \Psi(t), \quad (7)$$

where  $\Psi(t) = (\psi_1(t), \psi_2(t), \dots, \psi_n(t))^T$ ,  $G = (g_1, g_2, \dots, g_n)^T$  and  $g_i$  s are obtained by (5).

Also by block pulse series representation, we can approximate any arbitrary real bounded function  $g \in L^2([0, T])$  as follows:

$$g(t) \approx \sum_{i=1}^n g_i \psi_i(t) = G^T \Psi(t) = \Psi^T(t) G \quad (8)$$

As well as for any two variables function  $f(u, t) \in L^2([0, T_1] \times [0, T_2])$ , we have

$$f(u, t) = \Phi^T(u) F \Psi(t) = \Psi^T(t) F^T \Phi(u), \quad (9)$$

where  $\Phi(u)$  and  $\Psi(t)$  are  $n_1$  and  $n_2$  dimensional vectors of BFSs, respectively, and  $F$  is a  $n_1 \times n_2$  matrix with entries

$$f_{ij} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} f(u, t) \phi_i(u) \psi_j(t) dt du, \quad (10)$$

where  $h_1 = \frac{T_1}{n_1}$ ,  $h_2 = \frac{T_2}{n_2}$ . Also we set  $n_1 = n_2 = n$ .

### 3 Operational Matrices

Operational matrices have the special role in solving deterministic and stochastic integrals of BPF. Here, first in two subsections we review calculations of two operational matrices [8] and [10], for convenience name them constant and Wiener operational matrices, respectively. Then in the third subsection we solve the jump integral of BPF and present the Poisson operational matrix.

### 3.1 The Constant Operational Matrix

From [7], we can write

$$\int_0^t \psi_i(u)du = \begin{cases} 0 & 0 \leq t < (i-1)h, \\ t - (i-1)h & (i-1)h \leq t < ih, \\ h & ih \leq t < T. \end{cases}$$

With assumption  $t - (i-1)h = \frac{h}{2}$  at the mid-point of  $[(i-1)h, ih)$ , we can write

$$\int_0^t \psi_i(u)du \approx (0, \dots, 0, \frac{h}{2}, h, \dots, h)\Psi(t),$$

where  $\frac{h}{2}$  inserts in  $i$  th place of vector. Therefore,

$$\int_0^t \Psi(u)du \approx M_c(\Psi(t)), \tag{11}$$

where  $M_c$  is a constant operational matrix, as follows:

$$M_c = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}_{n \times n}.$$

So, for every function  $g(t)$

$$\int_0^t g(u)du \approx \int_0^t G^T \Psi(u)du \approx G^T P \Psi(t) \tag{12}$$

### 3.2 The Wiener Operational Matrix

For computing the Ito integral of BPFs we proceed as follows, [10]:

$$\int_0^t \psi_i(u)dW(u) = \begin{cases} 0 & 0 \leq t < (i-1)h \\ W(t) - W((i-1)h) & (i-1)h \leq t < ih \\ W(ih) - W((i-1)h) & ih \leq t < T \end{cases}$$

At the mid-point of  $[(i-1)h, ih)$ , we set  $W(t) - W((i-1)h) = W((i-0.5)h) - W((i-1)h)$ .

Then

$$\int_0^t \psi_i(u)dW(u) \approx (0, \dots, 0, W((i-0.5)h) - W((i-1)h), \\ W(ih) - W((i-1)h), \dots, W(ih) - W((i-1)h))\Psi(t),$$

where  $W((i-0.5)h) - W((i-1)h)$  is  $i$  th component. Therefore,

$$\int_0^t \Psi(u)dW(u) \approx M_w \Psi(t) \tag{13}$$

where  $M_w$  is the Wiener operational matrix, as follows:

$$M_w = \begin{pmatrix} W(h/2) & W(h) & W(h) & \dots & W(h) \\ 0 & W(3h/2) - W(h) & W(2h) - W(h) & \dots & W(2h) - W(h) \\ 0 & 0 & W(5h/2) - W(2h) & \dots & W(3h) - W(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & W((2n-1)h/2) - W((n-1)h) \end{pmatrix}_{n \times n}.$$

Thus,

$$\int_0^t g(u) dW(u) \approx \int_0^t G^T \Psi(u) dW(u) \approx G^T M_w \Psi(t), \tag{14}$$

for every function  $g(t)$ .

### 3.3 The Poisson Operational Matrix

We compute the jump integral of BFSs, as follows:

$$\int_0^t \int_{\mathbb{R}^+} \psi_i(u) \eta(z) \tilde{N}(du, dz) = \begin{cases} 0 & 0 \leq t < (i-1)h \\ \int_{\mathbb{R}^+} \eta(z) (\tilde{N}(t, dz) - \tilde{N}((i-1)h, dz)) & (i-1)h \leq t < ih \\ \int_{\mathbb{R}^+} \eta(z) (\tilde{N}(ih, dz) - \tilde{N}((i-1)h, dz)) & ih \leq t < T \end{cases}$$

Assuming

$$\int_{\mathbb{R}^+} \eta(z) (\tilde{N}(t, dz) - \tilde{N}((i-1)h, dz)) \approx \int_{\mathbb{R}^+} \eta(z) (\tilde{N}((i-0.5)h, dz) - \tilde{N}((i-1)h, dz))$$

at the mid-point of  $[(i-1)h, ih]$  we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^+} \psi_i(u) \eta(z) \tilde{N}(du, dz) &\approx (0, \dots, 0, \int_{\mathbb{R}^+} \eta(z) (\tilde{N}((i-0.5)h, dz) - \tilde{N}((i-1)h, dz)), \\ &\int_{\mathbb{R}^+} \eta(z) (\tilde{N}(ih, dz) - \tilde{N}((i-1)h, dz)), \dots, \\ &\int_{\mathbb{R}^+} \eta(z) (\tilde{N}(ih, dz) - \tilde{N}((i-1)h, dz))) \Psi(t), \end{aligned}$$

where  $\int_{\mathbb{R}^+} \eta(z) (\tilde{N}((i-0.5)h, dz) - \tilde{N}((i-1)h, dz))$  is  $i$  th member of the vector. Then

$$\int_0^t \int_{\mathbb{R}^+} \Psi(u) \eta(z) \tilde{N}(du, dz) \approx M_p \Psi(t), \tag{15}$$

where  $M_p$  is the Poisson operational matrix, as follows:

$$M_p = \begin{pmatrix} \int_{\mathbb{R}^+} \eta(z) \tilde{N}(h/2, dz) & \int_{\mathbb{R}^+} \eta(z) \tilde{N}(h, dz) & \int_{\mathbb{R}^+} \eta(z) \tilde{N}(h, dz) & \dots & \int_{\mathbb{R}^+} \eta(z) \tilde{N}(h, dz) \\ 0 & \int_{\mathbb{R}^+} \eta(z) (\tilde{N}(3h/2, dz) - \tilde{N}(h, dz)) & \int_{\mathbb{R}^+} \eta(z) (\tilde{N}(2h, dz) - \tilde{N}(h, dz)) & \dots & \int_{\mathbb{R}^+} \eta(z) (\tilde{N}(2h, dz) - \tilde{N}(h, dz)) \\ 0 & 0 & \int_{\mathbb{R}^+} \eta(z) (\tilde{N}(5h/2, dz) - \tilde{N}(2h, dz)) & \dots & \int_{\mathbb{R}^+} \eta(z) (\tilde{N}(3h, dz) - \tilde{N}(2h, dz)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \int_{\mathbb{R}^+} \eta(z) (\tilde{N}(((2n-1)h)/2, dz) - \tilde{N}((n-1)h, dz)) \end{pmatrix}_{n \times n}$$

So, for every function  $g(t)$

$$\int_0^t \int_{\mathbb{R}^+} g(u)\eta(z)\tilde{N}(du, dz) \simeq \int_0^t \int_{\mathbb{R}^+} G^T \Psi(u)\eta(z)\tilde{N}(du, dz) \simeq G^T M_p \Psi(t).$$

### 4 An Approximate Answer for Exponential Levy Equation

We apply results of previous sections to find an approximate answer for equation (1).

Using (8) and (9), we can write

$$\begin{aligned} S(t) &\simeq S^T \Psi(t) = \Psi^T(t)S, \\ r(t) &\simeq R^T \Psi(t) = \Psi^T(t)R, \\ \mu(u, t) &\simeq \Phi^T(u)\Gamma\Psi(t) = \Psi^T(t)\Gamma^T\Phi(u), \\ \sigma_i(u, t) &\simeq \Phi^T(u)\Sigma_i\Psi(t) = \Psi^T(t)\Sigma_i^T\Phi(u), \quad i = 1, \dots, n, \\ \zeta_i(u, t) &\simeq \Phi^T(u)\Xi_i\Psi(t) = \Psi^T(t)\Xi_i^T\Phi(u), \quad i = 1, \dots, n, \end{aligned} \tag{16}$$

where  $S$  and  $R$  are block pulse coefficient vectors. Also  $\Gamma$ ,  $\Sigma_i$  and  $\Xi_i$ ,  $i = 1, \dots, n$ , are block pulse coefficient matrices. Substituting (16) in equation (1), we have

$$\begin{aligned} S^T \Psi(t) &\simeq R^T \Psi(t) + S^T \left( \int_0^t \Phi(u)\Phi^T(u)du \right) \Gamma\Psi(t) + S^T \left( \sum_{i=1}^n \int_0^t \Phi(u)\Phi^T(u)dW_i(u) \right) \Sigma_i\Psi(t) \\ &\quad + S^T \left( \sum_{i=1}^n \int_0^t \int_{\mathbb{R}^+} \Phi(u)\Phi^T(u)\eta(z)\tilde{N}_i(du, dz) \right) \Xi_i\Psi(t). \end{aligned}$$

From (6) and (11), we get

$$\begin{aligned} \left( \int_0^t \Phi(u)\Phi^T(u)du \right) \Gamma\Psi(t) &= \left( \int_0^t \Psi(u)\Psi^T(u)du \right) \Gamma\Psi(t) = \begin{pmatrix} M_c^1 \Psi(t) \Gamma^1 \Psi(t) \\ M_c^2 \Psi(t) \Gamma^2 \Psi(t) \\ \vdots \\ M_c^n \Psi(t) \Gamma^n \Psi(t) \end{pmatrix} \\ &= \begin{pmatrix} M_c^1 D_{\Gamma^1} \\ M_c^2 D_{\Gamma^2} \\ \vdots \\ M_c^n D_{\Gamma^n} \end{pmatrix} \Psi(t) = \mathcal{C} \Psi(t), \end{aligned} \tag{17}$$

where  $\Gamma^j$  and  $M_c^j$ ,  $j = 1, 2, \dots, n$ , are the  $j$  th row of matrices  $\Gamma$  and  $M_c$ , respectively.  $D_{\Gamma^j}$  is a diagonal matrix which  $\Gamma^j$  s are its diagonal entries and

$$\mathcal{C} = \frac{h}{2} \begin{pmatrix} \gamma_{11} & 2\gamma_{12} & 2\gamma_{13} & \dots & 2\gamma_{1n} \\ 0 & \gamma_{22} & 2\gamma_{23} & \dots & 2\gamma_{2n} \\ 0 & 0 & \gamma_{33} & \dots & 2\gamma_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_{nn} \end{pmatrix}_{n \times n},$$

as well as  $\gamma_{ij}$  s are entries of the matrix  $\Gamma$ .

For the Ito integrals, we obtain

$$\left(\int_0^t \Phi(u)\Phi^T(u)dW(u)\right)\Sigma_i\Psi(t) = \left(\int_0^t \Psi(u)\Psi^T(u)dW(u)\right)\Sigma_i\Psi(t)$$

$$= \begin{pmatrix} M_w^1\Psi(t)\Sigma_i^1\Psi(t) \\ M_w^2\Psi(t)\Sigma_i^2\Psi(t) \\ \vdots \\ M_w^n\Psi(t)\Sigma_i^n\Psi(t) \end{pmatrix} = \begin{pmatrix} M_w^1D_{\Sigma_i^1} \\ M_w^2D_{\Sigma_i^2} \\ \vdots \\ M_w^nD_{\Sigma_i^n} \end{pmatrix} \Psi(t) = \mathcal{W}_i\Psi(t), \quad (18)$$

where  $M_w^j$  and  $\Sigma_i^j, j=1,2,\dots,n$ , are the  $j$ th row of matrices  $M_w$  and  $\Sigma_i$ , respectively.  $D_{\Sigma_i^j}$  is a diagonal matrix which  $\Sigma_i^j$ s are its diagonal entries and

$$\mathcal{W}_i = \begin{pmatrix} \sigma_i^1 W(h/2) & \sigma_i^2 W(h) & \sigma_i^3 W(h) & \dots & \sigma_i^n W(h) \\ 0 & \sigma_i^2 (W(3h/2)-W(h)) & \sigma_i^3 (W(2h)-W(h)) & \dots & \sigma_i^n (W(2h)-W(h)) \\ 0 & 0 & \sigma_i^3 (W(5h/2)-W(2h)) & \dots & \sigma_i^n (W(3h)-W(2h)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_i^n (W(((2n-1)h)/2)-W((n-1)h)) \end{pmatrix},$$

as well as  $\sigma_{ij}$ s are entries of the matrix  $\Sigma$ .

Also for the jump integrals, we can write

$$\left(\int_0^t \int_{\mathbb{R}^+} \Phi(u)\Phi^T(u)\eta(z)\tilde{N}(du,dz)\right)\Xi_i\Psi(t) = \left(\int_0^t \int_{\mathbb{R}^+} \Psi(u)\Psi^T(u)\eta(z)\tilde{N}(du,dz)\right)\Xi_i\Psi(t)$$

$$= \begin{pmatrix} M_p^1\Psi(t)\Xi_i^1\Psi(t) \\ M_p^2\Psi(t)\Xi_i^2\Psi(t) \\ \vdots \\ M_p^n\Psi(t)\Xi_i^n\Psi(t) \end{pmatrix} = \begin{pmatrix} M_p^1D_{\Xi_i^1} \\ M_p^2D_{\Xi_i^2} \\ \vdots \\ M_p^nD_{\Xi_i^n} \end{pmatrix} \Psi(t)$$

$$= \mathcal{P}_i\Psi(t), \quad (19)$$

where  $M_p^j$  and  $\Xi_i^j, j=1,2,\dots,n$ , are the  $j$ th row of matrices  $M_p$  and  $\Xi_i$ , respectively.  $D_{\Xi_i^j}$  is the diagonal matrix that  $\Xi_i^j$ s are its diagonal components and

$$\mathcal{P}_i = \begin{pmatrix} \xi_i^{11} \int_{\mathbb{R}^+} \eta(z)\tilde{N}(h/2,dz) & \xi_i^{12} \int_{\mathbb{R}^+} \eta(z)\tilde{N}(h,dz) & \xi_i^{13} \int_{\mathbb{R}^+} \eta(z)\tilde{N}(h,dz) & \dots & \xi_i^{1n} \int_{\mathbb{R}^+} \eta(z)\tilde{N}(h,dz) \\ 0 & \xi_i^{22} \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(3h/2,dz) - \tilde{N}(h,dz)) & \xi_i^{23} \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(2h,dz) - \tilde{N}(h,dz)) & \dots & \xi_i^{2n} \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(2h,dz) - \tilde{N}(h,dz)) \\ 0 & 0 & \xi_i^{33} \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(5h/2,dz) - \tilde{N}(2h,dz)) & \dots & \xi_i^{3n} \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(3h,dz) - \tilde{N}(2h,dz)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \xi_i^{nn} \int_{\mathbb{R}^+} \eta(z)(\tilde{N}(((2n-1)h)/2,dz) - \tilde{N}((n-1)h,dz)) \end{pmatrix}$$

as well as  $\xi_{ij}$ s are components of the matrix  $\Xi$ . Thus

$$S^T\Psi(t) \approx R^T\Psi(t) + S^T\mathcal{C}\Psi(t) + S^T\left(\sum_{i=1}^n \mathcal{W}_i\right)\Psi(t) + S^T\left(\sum_{i=1}^n \mathcal{P}_i\right)\Psi(t),$$

and

$$S^T(I - \mathcal{C} - \sum_{i=1}^n \mathcal{W}_i - \sum_{i=1}^n \mathcal{P}_i) \approx R^T.$$

Assuming  $A = (I - \mathcal{C} - \sum_{i=1}^n \mathcal{W}_i - \sum_{i=1}^n \mathcal{P}_i)^T$ , we obtain the following linear lower triangular system

$$AS \simeq R.$$

Finally, by the forward substitution method, we can easily solve the latter equation.

### 5 The Order of Convergence

The current section confirms that the convergence rate of the proposed method is  $O(h)$ .

**Theorem 1.** [10]: Suppose that  $\hat{g}(t) = \sum_{i=1}^n g_i \psi_i(t)$  is the block pulse representation for  $g(t) \in L^2[0,1)$ . Then  $\|E(t)\| = \|g(t) - \hat{g}(t)\| \leq \frac{h}{2\sqrt{3}} (\sup_{t \in [0,1]} \|g'(t)\|)$ .

**Theorem 2.** [10]: Suppose that  $\hat{f}(u, t) = \sum_{i=1}^n \sum_{j=1}^n f_{ij} \phi_i(u) \psi_j(t)$  is the block pulse representation for  $f(u, t) \in L^2([0,1) \times [0,1))$ . Then

$$\|E(u, t)\| = \|f(u, t) - \hat{f}(u, t)\| \leq \frac{h}{2\sqrt{3}} (\sup_{(x,y) \in T} \|f'_u(x, y)\|^2 + \sup_{(x,y) \in T} \|f'_t(x, y)\|^2)^{1/2},$$

where  $T = [0,1) \times [0,1)$ .

**Theorem 3:** Suppose that  $S(t)$  and  $\hat{S}(t)$  are exact and approximate solution of (1), respectively, as well as

$$\begin{aligned} \|S(t)\| &\leq \nu, \\ \|\mu(u, t)\| &\leq \rho, \\ \|\sigma_i(u, t)\| &\leq \chi_i, i = 1, \dots, n, \\ \|\zeta_i(u, t)\| &\leq \delta_i, i = 1, \dots, n, \\ \Lambda &< 1, \end{aligned}$$

for  $t \in [0,1)$  and  $(u, t) \in T = [0,1) \times [0,1)$ , where

$$\Lambda = \rho + V(h) + \sum_{i=1}^n (\chi_i + K_i(h)) \sup_{t \in [0,1)} \|W_i(t)\| + \sum_{i=1}^n (\delta_i + Q_i(h)) \sup_{t \in [0,1)} \|\int_{\mathbb{R}^+} \tilde{N}_i(t, dz)\|$$

Then

$$\|S(t) - \hat{S}(t)\| \leq \frac{\theta(h)}{1 - \Lambda}, \quad t \in [0,1)$$

where

$$\begin{aligned} \|S(t)\| &= (E[S^2(t)])^{1/2}, \\ N(h) &= \frac{h}{2\sqrt{3}} (\sup_{t \in [0,1)} \|r'(t)\|), \\ V(h) &= \frac{h}{2\sqrt{3}} (\sup_{(x,y) \in T} \|\mu'_u(x, y)\|^2 + \sup_{(x,y) \in T} \|\mu'_t(x, y)\|^2)^{1/2}, \\ K_i(h) &= \frac{h}{2\sqrt{3}} (\sup_{(x,y) \in T} \|\sigma'_{iu}(x, y)\|^2 + \sup_{(x,y) \in T} \|\sigma'_{it}(x, y)\|^2)^{1/2}, i = 1, \dots, n, \\ Q_i(h) &= \frac{h}{2\sqrt{3}} (\sup_{(x,y) \in T} \|\zeta'_{iu}(x, y)\|^2 + \sup_{(x,y) \in T} \|\zeta'_{it}(x, y)\|^2)^{1/2}, i = 1, \dots, n, \\ \theta(h) &= N(h) + V(h)\nu + \nu \sum_{i=1}^n K_i(h) \sup_{t \in [0,1)} \|W_i(t)\| + \nu \sum_{i=1}^n Q_i(h) \sup_{t \in [0,1)} \|\int_{\mathbb{R}^+} \tilde{N}_i(t, dz)\|. \end{aligned}$$



**Proof.**

Regarding to theorem 3 of [10], we can obtain  $\|S(t) - \hat{S}(t)\| = O(h)$ .

**6 Numerical Examples**

In this section, we check the accuracy of the presented approach by some numerical examples. The solution of SDE (1) is:

$$\begin{aligned}
 S(t) = & r(t) \exp\left(\int_0^t [\mu(u,t) - \frac{1}{2} \sum_{i=1}^k \sigma_i^2(u,t)] du + \sum_{i=1}^k \int_0^t \sigma_i(u,t) dW_i(u) \right. \\
 & + \sum_{i=1}^k \int_0^t \int_{\mathbb{R}^+} (\ln(1 + \eta(z)\zeta_i(u,t)) - \eta(z)\zeta_i(u,t)) \nu_i(dz) du \\
 & \left. + \sum_{i=1}^k \int_0^t \int_{\mathbb{R}^+} \ln(1 + \eta(z)\zeta_i(u,t)) \tilde{N}_i(du, dz)\right). \quad u, t \in [0,1]
 \end{aligned}$$

Since there is not an exact solution for Ito and jump integrals, we approximate them by the Simpson method. Then, we compare the final answer of the Simpson method with the solution of the presented method. Furthermore, we suppose that the jump component has the Gamma distribution and simulate it by the Poisson weighted algorithm of [21] in Matlab software.

**Example 1.** Let  $k = 1$ ,  $r(t) = 0.005$ ,  $\mu(u,t) = 0.04$ ,  $\sigma(u,t) = 0.02$ ,  $\eta(z) = z$ ,  $\zeta(u,t) = 0.02$ .

Table 1 shows the results of this example.  $M_e$  and  $S_e$  show the mean and the standard deviation of the error, respectively. Also, Fig. 1 depicts paths of approximate solutions of the stock price process by Simpson and the proposed methods.

**Table 1:** Numerical Solutions of Example 1 with  $m = 1000$  Iteration

n	$M_e$	$S_e$	$0.95CI_E$	
			Lower	Upper
10	3.09040e-04	7.48017e-05	3.04404e-04	3.13676e-04
25	5.09588e-04	8.19198e-05	5.04511e-04	5.14666e-04
40	6.45761e-04	8.65358e-05	6.40398e-04	6.51125e-04
55	7.54360e-04	8.44626e-05	7.49125e-04	7.59595e-04
70	8.54824e-04	8.51983e-05	8.49544e-04	8.60105e-04
85	9.44408e-04	9.33951e-05	9.38619e-04	9.50196e-04

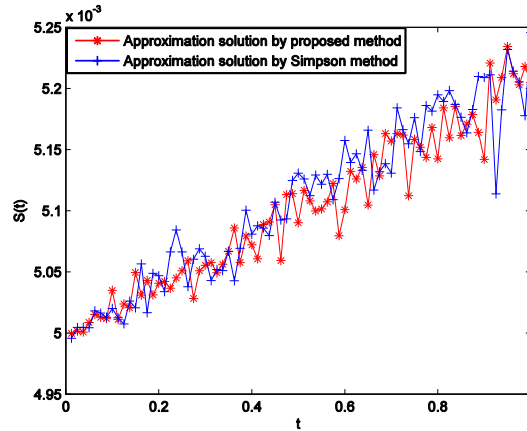


Fig. 1: Numerical Results of Example 1 with  $n = 80$  and  $m = 500$

**Example 2.** Let

$$k = 2, \quad r(t) = 0.01, \quad \mu(u, t) = u^2, \quad \sigma_1(u, t) = 0.1u^2, \quad \sigma_2(u, t) = 0.12u^3, \\ \eta_1(z) = z^2, \quad \eta_2(z) = z^4, \quad \zeta_1(u, t) = 0.02, \quad \zeta_2(u, t) = 0.04.$$

In two dimensions, the results are shown in Table 2 and Fig. 2.

**Table 2:** Numerical solutions of Example 2 with  $m = 1000$  iteration

n	$M_e$	$S_e$	$0.95CI_E$	
			Lower	Upper
10	1.69490e-03	2.26317e-03	1.55463e-03	1.83517e-03
25	2.35791e-03	4.32536e-03	2.08982e-03	2.62600e-03
40	2.84706e-03	2.97147e-03	2.66288e-03	3.03123e-03
55	3.50646e-03	5.07232e-03	3.19207e-03	3.82084e-03
70	4.08130e-03	9.79987e-03	3.47389e-03	4.68870e-03
85	7.84944e-03	2.63014e-02	6.21926e-03	9.47962e-03

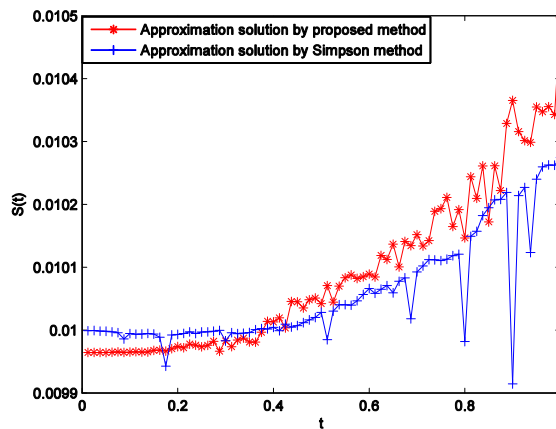


Fig. 2: Numerical Results of Example 2 with  $n = 80$  and  $m = 500$

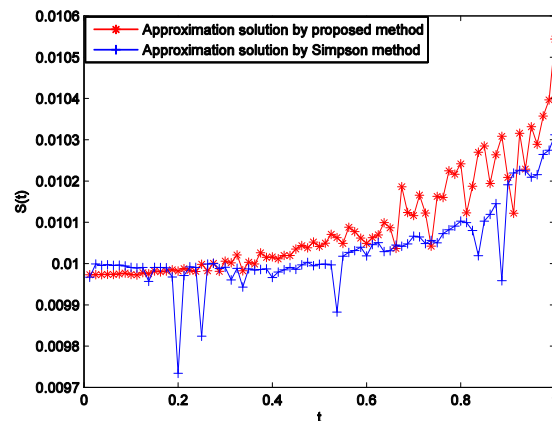
**Example 3.** Let

$$k = 3, \quad r(t) = 0.01, \quad \mu(u, t) = 0.1u^2, \quad \sigma_1(u, t) = 0.2u^2, \quad \sigma_2(u, t) = 0.15u^3, \\ \sigma_3(u, t) = 0.25u, \quad \eta_1(z) = z^2, \quad \eta_2(z) = z, \quad \eta_3(z) = z^4, \quad \zeta_1(u, t) = 0.02, \\ \zeta_2(u, t) = 0.04, \quad \zeta_3(u, t) = 0.07.$$

Table 3 and Fig. 3 reveal the numerical results of the current example in three dimensions.

**Table 3:** Numerical Solutions of Example 3 with  $m = 1000$  Iteration

n	$M_e$	$S_e$	$0.95CI_E$	
			Lower	Upper
10	3.53089e-03	1.77218e-03	3.42105e-03	3.64073e-03
25	5.73899e-03	1.68289e-03	5.63468e-03	5.84329e-03
40	7.20607e-03	1.98405e-03	7.08309e-03	7.32904e-03
55	8.57532e-03	2.38263e-03	8.42765e-03	8.72300e-03
70	9.85402e-03	2.61912e-03	9.69169e-03	1.00163e-02
85	1.09557e-02	3.10986e-03	1.07630e-02	1.11485e-02



**Fig. 3:** Numerical Results of Example 3 with  $n = 80$  and  $m = 500$

## 7 Conclusions

Multidimensional exponential Levy equations arise in many hedging and pricing problems. Unfortunately, there is not an exact solution for such stochastic equations and also their computational rate is relatively high. Thus, we need numerical methods to approximate the answer. For this purpose, we apply block pulse functions (BPFs) as basic functions and obtain the Poisson operational matrix for the jump integral of each BPF. We show that these functions can convert our equation to a linear lower triangular system by operational matrices. Then via the forward substitution method, we get an approximate answer with the convergence rate of  $O(h)$ . Finally, we check the accuracy of our results on some examples.

## References

- [1] Merton, R.C., *Option pricing when underlying stock returns are discontinuous*, Journal of Financial Economic, 1976, **2**, P. 125-144.
- [2] Kou, S. G., *A jump diffusion model for option pricing*, Management Science, 2002, **48**, P. 1086–1101.
- [3] Glasserman, P. and Kou, S.G., *The term structure of simple forward rates with jump risk*, Mathematical finance, 2003, **13**(3), P. 383–410.
- [4] Cont, R. and Tankov, P., *Financial Modelling with Jump Processes*, Financial Mathematics Series, Chapman and Hall/CRC, 2004.
- [5] Brummelhuis, R., Chan, R.T.L., *A radial basis function scheme for option pricing in exponential Levy models*, Applied Mathematical Finance, 2014, **21**(3). P. 238-269.
- [6] Chen, X. and Wan, J., *Option pricing for Time-change Exponential levy Model under MEMM*, Acta Mathematicae Applicatae Sinica, English Series, 2007, **23**(4). P. 651-664.
- [7] Izadikhah, M., *Improving the Banks Shareholder Long Term Values by Using Data Envelopment Analysis Model*, Advances in Mathematical Finance and Applications, 2018, **3**(2), P. 27-41. Doi: 10.22034/AMFA.2018.540829
- [8] Jiang, Z.H. and Schaufelberger, W., *Block Pulse Functions and Their Applications in Control Systems*, Springer-Verlag, 1990.
- [9] Prasada Rao, G., *Piecewise Constant Orthogonal Functions and their Application to Systems and Control*, Springer, Berlin, 1983.
- [10] Maleknejad, K., Khodabin, M., Rostami, M., *A numerical method for solving m-dimensional stochastic Ito-Volterra integral equations by stochastic operational matrix*, Computers and Mathematics with Applications, 2012, **63**, P. 133–143.
- [11] Maleknejad, K., Khodabin, M., Rostami, M., *Numerical Solution of Stochastic Volterra Integral Equations By Stochastic Operational Matrix Based on Block Pulse Functions*, Mathematical and Computer Modeling, 2012, **55**, P. 791–800.
- [12] Khodabin, M., Maleknejad, K., Rostami, M., *Numerical approach for solving stochastic Volterra-Fredholm integral equations by stochastic operational matrix*, Computers and Mathematics with Applications, 2012, **64**, P. 1903–1913.
- [13] Maleknejad, K., Sohrabi, S., Rostami, Y., *Numerical solution of nonlinear Volterra integral equations of the second kind by using Chebyshev polynomials*, Applied Mathematics and Computation, 2007, **188**, P. 123–128.
- [14] Maleknejad, K., Rahimi, B., *Modification of block pulse functions and their application to solve numerically Volterra integral equation of the first kind*, Communications in Nonlinear Science and Numerical Simulation 2011, **16**, P. 2469–2477.
- [15] Karapinar, E., Mosai, S., Taherinejad, F., *A Corporate Perspective on Effect of Asymmetric Verifiability on Investors' Expectation Differences*, Advances in Mathematical Finance and Applications, 2019, **4**(4), P.1-18. Doi: 10.22034/amfa.2019.1869165.1227

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- [16] Maleknejad, K., Safdari, H., Nouri, M., *Numerical solution of an integral equations system of the first kind by using an operational matrix with block pulse functions*, International Journal of Systems Science, 2011, **42**, P. 195–199.
- [17] Zhang, X., *Stochastic Volterra equations in Banach spaces and stochastic partial differential equation*, Journal of Functional Analysis, 2010, **258**, P. 1361–1425.
- [18] Manafian, J. and Bolghar, P., *Numerical solutions of nonlinear 3-dimensional Volterra integral-differential equations with 3D-block-pulse functions*, Mathematical Methods in the Applied Sciences, 2018, **41** (12), P. 4867-4876.
- [19] Safavi, M., Khajehnasiri, A.A., Angelamaria Cardone, A., *Numerical solution of nonlinear mixed Volterra-Fredholm integro-differential equations by two-dimensional block-pulse functions*, Cogent Mathematics and Statistics, 2018, **2** (1), P. 1-13.
- [20] Yonthanthum, W., Rattana, A., Razzaghi, M., *An approximate method for solving fractional optimal control problems by the hybrid of block-pulse functions and Taylor polynomials*, Optimal Control Applications and Methods, 2018, **39** (2), P. 873-887.
- [21] Lee, J. and Kim, Y., *Sampling methods for NTR processes*, in Proceedings of International Workshop for Statistics (SRCCS '04), Lecture Note, Seoul, Korea, 2004.