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On Vector Equilibrium Problem with Generalized Pseudomonotonicity

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ABSTRACT

In this paper, first a short history of the notion of equilibrium problem in Economics and Nash game theory is stated. Also the relationship between equilibrium problem and important mathematical problems like optimization problem, nonlinear programming, variational inequality problem, fixed point problem and complementarity problem are given. The concept of generalized pseudomonotonicity for vector valued bifunctions is introduced and by using it some existence results for the vector equilibrium problem, in the setting of topological vector spaces, are presented. Some examples in order to illustrate the main results and compare them with the corresponding published results are furnished. Further, the compactness of the solution set of vector equilibrium problem is investigated.

1 Introduction

In economics, economic equilibrium is a state where economic forces such as supply and demand are balanced and in the absence of external influences the (equilibrium) values of economic variables will not change. For example, in the standard textbook model of perfect competition, equilibrium occurs at the point at which quantity demanded and quantity supplied is equal. Market equilibrium [20, 7] in this case refers to a condition where a market price is established through competition such that the amount of goods or services sought by buyers is equal to the amount of goods or services produced by sellers. This price is often called the competitive price or market clearing price and will tend not to change unless demand or supply changes, and the quantity is called "competitive quantity" or market clearing quantity. However, the concept of equilibrium in economics also applies to imperfectly competitive markets, where it takes the form of a Nash equilibrium.

1.1 Properties of Equilibrium

Three basic properties of equilibrium in general have been proposed by Dixon (see, Chapter13 of [6]) as follows:

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Equilibrium property P_1: The behaviour of agents is consistent.
Equilibrium property P_2: No agent has an incentive to change its behaviour.
Equilibrium Property P_3: Equilibrium is the outcome of some dynamic process (stability).

1.2 Example: Competitive Equilibrium

In a competitive equilibrium, supply equals demand. Property P_1 is satisfied ,because at the equilibrium price the amount supplied is equal to the amount demanded. Property P_2 is also satisfied. Demand is chosen to maximize utility given the market price: no one on the demand side has any incentive to demand more or less at the prevailing price. Likewise supply is determined by firms maximizing their profits at the market price: no firm will want to supply any more or less at the equilibrium price. Hence, agents on neither the demand side nor the supply side will have any incentive to alter their actions. To see whether Property P_3 is satisfied, consider what happens when the price is above the equilibrium. In this case there is an excess supply, with the quantity supplied exceeding that demanded. This will tend to put downward pressure on the price to make it return to equilibrium. Likewise where the price is below the equilibrium point there is a shortage in supply leading to an increase in prices back to equilibrium. Not all equilibria are "stable" in the sense of Equilibrium property P_3 It is possible to have competitive equilibria that are unstable. However, if an equilibrium is unstable, it raises the question of how you might get there. Even if it satisfies properties P_1 and P_2, the absence of P_3means that the market can only be in the unstable equilibrium if it starts off there. In most simple microeconomic stories of supply and demand a static equilibrium is observed in a market; however, economic equilibrium can be also dynamic. Equilibrium may also be economy-wide or general, as opposed to the partial equilibrium of a single market. Equilibrium can change if there is a change in demand or supply conditions. For example, an increase in supply will disrupt the equilibrium, leading to lower prices. Eventually, a new equilibrium will be attained in most markets. Then, there will be no change in price or the amount of output bought and sold until there is an exogenous shift in supply or demand (such as changes in technology or tastes). That is, there are no endogenous forces leading to the price or the quantity.

1.3 Example: Nash Equilibrium

The Nash equilibrium is widely used in economics as the main alternative to competitive equilibrium. It is used whenever there is a strategic element to the behaviour of agents and the "price taking" assumption of competitive equilibrium is inappropriate. The first use of the Nash equilibrium was in the Cournot duopoly as developed by Cournot in his book [5]. Both firms produce a homogenous product: given the total amount supplied by the two firms, the (single) industry price is determined using the demand curve. This determines the revenues of each firm (the industry price times the quantity supplied by the firm). The profit of each firm is then this revenue minus the cost of producing the output. Clearly, there is a strategic interdependence between the two firms. If one firm varies its output, this will in turn affect the market price and so the revenue and profits of the other firm. We can define the payoff function which gives the profit of each firm as a function of the two outputs chosen by the firms. Cournot [5] assumed that each firm chooses its own output to maximize its profits given the output of the other firm. The Nash equilibrium occurs when both firms are producing the outputs which maximize their own profit given the output of the other firm. In terms of the equilibrium properties, we can see that P_2 is satisfied: in a Nash equilibrium, neither firm has an incentive to deviate from the Nash equilibrium given the output of the other firm. P_1 is satisfied since the payoff function ensures that the market price is consistent with the outputs supplied and that each firms profits equal revenue minus cost at this output.

1.4 Dynamic Equilibrium

Whereas in a static equilibrium all quantities have unchanging values, in a dynamic equilibrium various quantities may all be growing at the same rate, leaving their ratios unchanging. For example, in the neoclassical growth model, the working population is growing at a rate which is exogenous (determined outside the model, by non-economic forces). In dynamic equilibrium, output and the physical capital stock also grow at that same rate, with output per worker and the capital stock per worker unchanging. Similarly, in models of inflation a dynamic equilibrium would involve the price level, the nominal money supply, nominal wage rates, and all other nominal values growing at a single common rate, while all real values are unchanging, as is the inflation rate [13,17,19]. The process of comparing two dynamic equilibria to each other is known as comparative dynamics. For example, in the neoclassical growth model, starting from one dynamic equilibrium based in part on one particular saving rate, a permanent increase in the saving rate leads to a new dynamic equilibrium in which there are permanently higher capital per worker and productivity per worker, but an unchanged growth rate of output; so it is said that in this model the comparative dynamic effect of the saving rate on capital per worker is positive but the comparative dynamic effect of the saving rate on the output growth rate is zero. Equilibrium problems of theoretical aspect and its applications in Economics, optimization, Fixed point and so on (see, for instance, [4]) recently attracts increasing attention and are proven to be significant in the study of optimization, variational inequalities and complementarity problems. The formulation of such problems follows. Given a set K and a bifunction $F : K \times K \to \mathbb{R}$. An (scalar) equilibrium problem (EP) consists of finding $x \in K$ such that

$$F(x, y) \geq 0, \qquad \forall y \in K .$$

Many problems of practical interests involve an equilibrium problem formulation in their description, see for example [2, 4]. Most results on the existence of solutions for equilibrium problems are guaranteed by special algebraic properties on the bifunction F, known as generalized monotonicity, see e.g. [4, 11]. Economists use the term equilibrium to describe the balance between supply and demand in the marketplace. Under ideal market conditions, price tends to settle within a stable range when output satisfies customer demand for that good or service. Equilibrium is vulnerable to both internal and external influences. The appearance of a new product that disrupts the marketplace, such as the iPhone, is one example of an internal influence. The collapse of the real estate market as part of the Great Recession is an example of an external influence. The motivation of introducing equilibrium problem and existence theorems was in order to describe them by applying mathematical models and methods. In this article we deal with the vector equilibrium problem and present some existence theorems by using a new definition of generalized psudomonotonicity. The equilibrium price and quantity in a market are located at the intersection of the market supply curve and the market demand curve.

Often, economists must churn through massive amounts of data to solve equilibrium equations. This step-by-step guide will walk you through the basics of solving such problems. Throughout this paper, let Ybe a (Hausdorff) topological vector space and C denote a closed and convex cone of Y. The set C is pointed if $C \cap (-C) = 0$. Also it is called a solid cone when $intC \neq \emptyset$, where $intC$ stands for the topological interior of C. The convex cone C induces a preorder on Y as follows:

$$y_1 \preceq y_2 \Leftrightarrow y_2 - y_1 \epsilon C .$$

Notation: $y_1 \precneqq y_2 \Leftrightarrow y_2 - y_1 \epsilon C \setminus \{0\}$.

In addition, if C is a pointed cone then the binary relation $\preceq$ is a partial ordering. Likewise, we write $y_1 \prec y_2$ if $y_2 - y_1 \epsilon intC$.

Let X be a topological vector space and let $K \subset X$ be nonempty and closed convex set. The vector equilibrium problem (for short, (VEP)) for the bifunction $f : K \times K \rightarrow Y$ is to find $\bar{x} \epsilon K$, such that

$$f(\bar{x}, y) \nprec 0, \forall y \epsilon K, \tag{1}$$

with $f(x,x) = 0, \forall x \epsilon K$.

2 Preliminaries

Throughout the paper, unless otherwise specified, K is a nonempty closed convex subset of a topological vector space X and $(Y ; C)$ is an ordered topological vector space induced by the pointed closed convex cone C with $intC$ nonempty and $C^* = \{y^* \epsilon Y^* : \langle y^*, x \rangle \geq 0, \forall x \epsilon C\} \neq \emptyset$. The following definition and lemma will be useful in article. Note that from the bipolar theorem (see [18]) we have

$$y \epsilon C \Leftrightarrow [\langle y^*, y \rangle \geq 0, \forall y^* \epsilon C^*], \tag{2}$$

$$y \epsilon intC \Leftrightarrow \left[\langle y^*, y \rangle > 0, \forall y^* \epsilon\, C_+^* = \left\{y^* \epsilon Y^* : \langle y^*, y \rangle > 0, \forall x \in C \backslash \{0\}\right\}\right]. \tag{3}$$

Definition 2.1.([18]) The mapping $g : K \rightarrow Y$ is called C −convex if for every $x, x' \in K$ and $t \in [0,1]$ one has $tg(x) + (1-t)g(x') - g(tx + (1-t)x') \in C$.

Definition 2.2.([18]) Let X be a topological space. The function $f : X \rightarrow \mathbb{R}$ is lower semicontinuous at x_0 if

$$f(x_0) \leq \lim_{\alpha} inf f(x_\alpha).$$

for any net $\{x_\alpha\} \subset X$ such that $x_\alpha \rightarrow x_0$. Similarly, f is upper semicontinuous if and only if $-f$ is lower semicontinuous.

In the following we recall the definition of C−lower and C −upper semicontinuous for vector valued mappings.

Definition 2.3.([18]) Let X be a topological space and Y a topological vector space with $C \subset Y$ a solid convex cone. The vector valued mapping $g : X \rightarrow Y$ is called

(a) C −lower semicontinuous on X if for each fixed $x \epsilon X$ and for any $y \in intC$, there exists a neighbourhood $U(x)$ such that

$$g(x) \epsilon\, g(u) + y - intC, \quad \forall u \in U(x).$$

(b) C −upper semicontinuous on X if and only if for each $x \in X$ and for any $y \in intC$, there exists a neighbourhood $U(x)$ such that

$$g(u) \in g(x) + y - intC, \forall u \in U(x).$$

We need the following lemma in the next section.

Lemma 2.1. If $g : K \to Y$ is a $C-$lower semicontinuous mapping, then the set $A := \{x \in K, g(x) \notin intC\}$ is a closed set in K.

Proof. Suppose that $x \notin A$. Then $g(x) \in intC$. Hence by Definition 2.3 (a) there exists U(x) such that

$$g(x) \in g(u) + g(x) - intC, \forall u \in U(x),$$

which implies that $g(u) \in intC$. Therefore $U(x) \subset K \setminus A$. This shows that A is closed in K.■

Definition 2.4.([3]) A mapping $f : K \times K \to Y$ is said to be generalized pseudomonotone if there exists function $\alpha: X \times X \to Y$ with

$$\lim_{t\to 0} \frac{\alpha(ty + (1-t)x, x)}{t} \preceq 0,$$

such that the following implication holds:

$$f(x,y) \nprec 0 \Rightarrow f(y,x) \preceq \alpha(y,x), \forall x, y \epsilon K. \quad (4)$$

Remark 2.1. (i) Note that if in Definition 2.4 we take $\alpha(x,y) = 0$ for all $x, y \in X$ then the generalized pseudomonotone reduces to the $C-$pseudomonotonicity given in [10].

(ii) Also if f is generalized pseudomonotonicity with $\alpha(x,y) \preceq 0$; for all $x, y \in X$ then f is $C-$pseudomonotone. Indeed if $f(x,y) \succeq 0$ then $f(x,y) \in C$ and so $f(x,y) \nprec 0$ (note C is a pointed cone). Now the assertion follows from the generalized pseudomonotonicity with $\alpha(x,y) \preceq 0$,for all $x, y \in X$.

(iii) Definition 3.4 of [16] is a special case of Definition 2.4.

Definition 2.5. ([15])The mapping $f : X \times X \to Y$ is said to be weakly generalized pseudomonotone, if there exists a function $\alpha: X \to Y$ with $\alpha(tx) = k(t)\alpha(x)$ for all $t > 0$ and $x \in X$, where k is function from $(0, +\infty)$ to $(0, +\infty)$ with $\lim_{t\to 0} \frac{k(t)}{t} = 0$ such that for every pair of points $x, y \in K$, we have

$$f(x,y) \nprec 0 \Rightarrow f(x,y) \preceq \alpha(y-x) \quad (5)$$

Example 2.1. Let $X = Y = \mathbb{R}, K = [0,1], C = [0, +\infty]$, and $f(x,y) = x - y$. Then it is easy to verify that f is weakly generalized pseudomonotone mapping with $\alpha(x) = x^2 \; \forall x \in X$.

Remark that if f is weakly generalized pseudomonotone then it is generalized pseudomonotone. Because we can take $\alpha_1(x,y) = \alpha(x-y)$, for all $x, y \in X$. Then

$$\lim_{t\to 0} \frac{\alpha_1(x + t(y-x), x)}{t} = \lim_{t\to 0} \frac{k(t)\alpha(y-x)}{t} = 0.$$

Now the proof of the assertion follows from the relation (5).

Consequently, we have the following facts:

$C-$pseudomonotonicity ⇒ weakly generalized pseudomonotonicity ⇒ generalized pseudomonotonicity.

The following example shows that there are generalized pseudomonotone mappings which are not $C-$pseudomonotone.

Example 2.2. Let $X = \mathbb{R}$, K is any nonempty closed convex subset of $\mathbb{R}$, $Y = \mathbb{R}^2$,

$C = \{(x,y), x \geq 0, y \geq 0\}$, and

$$f(x,y) = ((x-y)^2, (x-y)^2).$$

Then f is generalized pseudomonotone mapping with

$$\alpha(x,y) = (2(x-y)^2, 2(x-y)^2)\ , \forall x, y \in X.$$

It is easy to see that f is not C −pseudomonotone.

Definition 2.6.([8]) Let K be a convex subset of a topological vector space E. A real valued mapping $f: K \to \mathbb{R}$ is called hemicontinuous if, for all $x, y \epsilon K$ the mapping $F: [0,1] \to X$ defined by $F(t) = f(tx + (1-t)y)$ is continuous at 0 from the right(that is $F(0) = \lim_{t \to 0^+} F(t)$).

Definition 2.7. ([8]) Let K be a nonempty subset of a topological vector space E. A mapping $F: K \to 2^E$ is said to be a KKM mapping, if for any finite subset $\{x_1, x_2, \dots, x_n\} \subset K$; the following inclusion holds:

$$co\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i).$$

The following results play crucial roles in the next section.

Lemma 2.2. ([8]) Let K be a nonempty subset of a Hausdorff topological vector space E and let $f: K \to 2^E$ be a KKM mapping. If $f(y)$ is closed in E for all $y \in E$ and compact for some $y \in K$ then $\bigcap_{y \in K} f(y) \neq \emptyset$.

Proposition 2.3. ([14]) Let f be a single-valued mapping from X into Y and $u^* \in C_+^*$. Let $\emptyset: X \to \mathbb{R}$ be a mapping defined by $\emptyset(x) = \langle u^*, g(x) \rangle$ for all $x \in X$.Then the following assertions are valid:

If f is C −lower semicontinuous (rep., C −upper semicontinuous) then ϕ is u.s.c:(resp., l.s.c).

3 Main Results

In this section we give some existence results of a solution for the VEP by applying generalized pseudomonotonicity. Moreover, under suitable assumptions the compactness of the solution set of VEP is investigated. The results of this part improve and generalize the corresponding results appeared in [1, 3, 9, 10, 15, 16].

Theorem 3.1. Let K be a nonempty closed convex subset of a topological vector space X and C be a solid pointed closed convex cone of a topological vector space Y.Suppose $f: K \times K \to Y$ is hemicontinuous in the first argument and generalized pseudomonotone. Let the following condition hold:

(a) For each $z \in K$,the mapping $x \mapsto f(z,x)$ is C −convex.

Then $\bar{x} \in K$ is a solution of VEP if and only if

$$f(y, \bar{x}) \preceq \alpha(y, \bar{x}), \forall y \in K. \quad (6)$$

Proof. Assume that $\bar{x} \in K$ is a solution of V EP (1) , i.e., $f(\overline{x}, y) \nless 0$, $\forall y \in K$ Since f is generalized pseudomonotone, we have

$$f(y, \bar{x}), \forall y \in K,$$

and so $\overline{x}$ is a solution of (6).

Conversely, suppose there exists an $\bar{x} \in K$ satisfying (6) Choose any point $y \in K$ and consider $x_t = ty + (1-t)\bar{x}$, $t \in [0,1]$. We have

$f(x_t, \bar{x}) \preceq \alpha(x_t, \bar{x}).$ (7)

Now condition (a) implies,

$$0 \preceq f(x_t, x_t) \preceq t f(x_t, y) + (1-t) f(x_t, \bar{x}),$$

which implies that

$t[f(x_t, \bar{x}) - f(x_t, y)] \preceq f(x_t, \bar{x}).$ (8)

From (7) and (8), we get

$$f(x_t, \bar{x}) - f(x_t, y) \preceq \frac{\alpha(x_t, y)}{t}, \forall y \in K.$$

Hence it follows from the hemicontinuity of f in the first argument, C is closed, and the property of α by taking $t \to 0$, we get $f(\bar{x}, y) \succeq 0$, which implies that $f(\bar{x}, y) \nprec 0 \ \forall y \in K$ (note C is pointed). Then $\overline{x}$ is a solution of VEP. This completes the proof.■

Note that if in Theorem 3.1 we take $Y = \mathbb{R}$ and $\alpha = 0$ then Theorem 3.1 collapses to Lemma 2.4 of [12]. Further Theorem 3.1 is a vector version of Lemma 2.1 of [11] and Lemma 2.4 of [12].

Theorem 3.2. Let K be a nonempty closed convex subset of a topological vector space X and C be a solid pointed closed convex cone of a topological vector space Y. Let $f : K \times K \to Y$ be hemicontinuous in the first argument and generalized pseudomonotone. Assume that the following conditions hold:

(i) for each $z \in K$; the mapping $x \mapsto f(z, x)$ is C −convex and C −lower semicontinuous;

(ii) $\exists y_0 \in K$ such that $\{x \in K; f(y_0, x) \preceq \alpha(y_0, x)\}$ is compact;

(iii) for each $z \in K$, the mapping $x \mapsto \alpha(z, x)$ is C −upper semicontinuous.

Then the solution set of VEP is nonempty and compact.

Proof. Consider the set valued mappings $F: K \to 2^X$ and $G: K \to 2^Y$ such that

$$F(y) = \{x \in K: f(x, y) \nprec 0\}, \forall y \in K,$$

and

$$G(y) = \{x \in K : f(y, x) \preceq \alpha(y, x)\}, \forall y \in K.$$

Now $\bar{x}$ solves VEP if and only if $\bar{x} \in \bigcap_{y \in K} F(y)$. Thus it suffices to prove $\bigcap_{y \in K} F(y) \neq \emptyset$. First we claim that F is a KKM mapping. Otherwise there exists $\{x_1, x_2, \dots, x_m\} \subset K$ such that $\{x_1, x_2, \dots, x_m\} \not\subseteq \bigcup_{i=1}^m F(x_i)$. Hence there exists $x_0 \in co\{x_1, x_2, \dots, x_m\}, x_0 = \sum_{i=1}^m t_i x_i$ where $t_i \geq 0$, $i = 1,2, \dots, m, \sum_{i=1}^m t_i = 1$, but $x_0 \notin \bigcup_{i=1}^m F(x_i)$. Then $f(x_0, x_i) \prec 0$; for $i = 1,2, \dots, m$. From (i) ,it follows that

$$0 \preceq f(x_0, x_0) \preceq \sum_{i=1}^{m} t_i f(x_0, x_i) \prec 0,$$

which is a contradiction and so F is a KKM mapping. The generalized pseudomonotonicity of f, implies that $F(y) \subset G(y), \forall y \in K$.Therefore G is also a KKM mapping. We prove that G(y) is a closed set for all $y \in K$. Suppose that $x_\alpha \epsilon\, G(y)$ be a net $x_\alpha \to x$. Then

$$f(y, x_\alpha) \preceq \alpha(y, x_\alpha).$$

Now suppose that $u^* \in C^*$ be an arbitrary element of C^*. Hence it follows from Proposition 2.3 that

$\langle u^*, f(y, x)\rangle \leq \lim_{\alpha} inf\, \langle u^*, f(y, x_\alpha)\rangle \leq \lim_{\alpha} sup\, \langle u^*, \alpha(y, x_\alpha)\rangle \leq \langle u^*, \alpha(y, x)\rangle$ (9)

then by (2) we have $\alpha(y,x) - f(y,x) \in C$.This means $f(y,x) \preceq \alpha(y,x)$. Consequently, $x \in G(y)$ and so $G(y)$ is a closed set. Then G satisfies all the assumptions of Lemma 2.2, note $G(y_0)$ is compact, and so $\bigcap_{y\in K} G(y) \neq \emptyset$. On the other hand Theorem 3.1 implies

$$\bigcap_{y\in K} F(y) = \bigcap_{y\in K} G(y).$$

So there exists $\overline{x} \in K$ such that $f(\overline{x}, y) \nless 0, \forall y \in K$, i.e VEP (1) has a solution.
Also since $\bigcap_{y\in K} F(y)$ is a closed set and $\bigcap_{y\in K} F(y) \subset G(y_0)$ then the solution set of VEP is a compact set.■

Theorem 3.3. Let K be a nonempty subset of a topological vector space X and (Y, C) is an ordered topological vector space induced by the pointed closed convex cone C with $intC \neq \emptyset$. Suppose $f : K \times K \to Y$ be hemicontinuous in the first argument and generalized pseudomonotone. Let the following conditions hold:

(i) $f(.,y)$ is C −lower semicontinuous, for all $y \in K$,

(ii) there exist a compact convex subset B of K such that
$\forall x \in K\backslash B, \exists z \in B$ such that $f(z,x) \succ \alpha(z,x)$.

Then the solution set of (VEP) is nonempty and compact.

Proof. Consider the set valued mappings $F : K \to 2^X$, and $G: K \to 2^X$ such that

$$F(y) = \{x \in K: f(x,y) \nless 0\}, \forall y \in K,$$

and

$$G(y) = \{x \in K: f(y,x) \preceq \alpha(y,x)\}, \forall y \in K.$$

Now $\bar{x}$ solves (VEP) if and only if $\bar{x} \in \bigcap_{y\in K} F(y)$. We note that for each $y \in K$ the set

$$F(y) = \{x \in K, f(x,y) \notin -intC\},$$

is a closed set, see Lemma 2.1. Now it follows from a similar method as given for the proof of Theorem 3.2 that VEP has a solution. We claim that

$$\bigcap_{y\in K} F(y) \subset B.$$

Otherwise, there exists

$$x \in \left(\bigcap_{y\in K} F(y)\right)\backslash B.$$

Then by (ii) of the hypothesis there exists $z \in K$ such that

$$f(z, x_0) \succ \alpha(z,x),$$

and so it follows from the generalized pseudomonotonocity of f that $f(x;\ z) \prec 0$, which is contradicted by $x \in \bigcap_{y\in K} F(y)$ This completes the proof.■

4 Conclusions

In this paper, a short history of the notion of equilibrium problem in Economics and Nash game theory was stated and the relationship between equilibrium problem and important mathematical problems were given. Some examples in order to illustrate the main results and compare them with the corresponding published results were presented.

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